



Contents lists available at ScienceDirect

## Journal of Financial Economics

journal homepage: [www.elsevier.com/locate/jfec](http://www.elsevier.com/locate/jfec)Informed traders and limit order markets<sup>☆</sup>Ronald L. Goettler<sup>a,\*</sup>, Christine A. Parlour<sup>b</sup>, Uday Rajan<sup>c</sup><sup>a</sup> Booth School of Business, University of Chicago, USA<sup>b</sup> Haas School of Business, University of California, Berkeley, USA<sup>c</sup> Ross School of Business, University of Michigan, Ann Arbor, USA

## ARTICLE INFO

## Article history:

Received 1 January 2008

Received in revised form

17 June 2008

Accepted 6 August 2008

Available online 5 April 2009

## JEL classification:

G14

C63

C73

D82

## Keywords:

Limit order market

Informed traders

Endogenous information acquisition

Computational game

## ABSTRACT

We consider a dynamic limit order market in which traders optimally choose whether to acquire information about the asset and the type of order to submit. We numerically solve for the equilibrium and demonstrate that the market is a “volatility multiplier”: prices are more volatile than the fundamental value of the asset. This effect increases when the fundamental value has high volatility and with asymmetric information across traders. Changes in the microstructure noise are negatively correlated with changes in the estimated fundamental value, implying that asset betas estimated from high-frequency data will be incorrect.

© 2009 Published by Elsevier B.V.

## 1. Introduction

Many financial markets around the world, including the Paris, Stockholm, Shanghai, Tokyo, and Toronto stock exchanges, are organized as limit order books. In addition, aspects of a limit order book are also incorporated into markets such as Nasdaq and the NYSE. In spite of the dominance of this market form, there is no dynamic model of information-based trade in which investors can choose to submit either market or limit orders.

Understanding dynamic order choice under asymmetric information is important because rational agents use different trading strategies for different assets. The characteristics of an asset (such as the volatility of changes in the fundamental value of the asset) affect whether agents acquire information about the asset, which in turn affects the trading strategies they employ, and thus the relationship between the fundamental value of an asset and its price or other market observables. Specifically, the information content in a limit order book differs depending on whether informed agents submit limit orders and the prices at which they do so. Therefore, to infer the fundamental characteristics of an asset from market observables, or the information content in observables, it is important to understand how agents' trading behavior differs across assets. We conduct a systematic study of a limit order market with asymmetric information to determine the effect of asset characteristics on trading behavior and market outcomes.

<sup>☆</sup> We have benefited greatly from comments by an anonymous referee and Elena Asparouhova, Kerry Back, Ekkehart Boehmer, Dmitry Livdan, Bill Lovejoy, Ben Van Roy, and seminar participants at Aladdin, CFS (Elthville), Michigan, Carnegie Mellon, Stanford, Texas A&M, HKUST Symposium, Oxford Summer Symposium, Utah WFC, INFORMS, NBER Market Microstructure, and WFA meetings.

\* Corresponding author. Tel.: +1773 702 7549.

E-mail addresses: [ronald.goettler@chicagobooth.edu](mailto:ronald.goettler@chicagobooth.edu) (R.L. Goettler), [parlour@haas.berkeley.edu](mailto:parlour@haas.berkeley.edu) (C.A. Parlour), [urajan@umich.edu](mailto:urajan@umich.edu) (U. Rajan).

In our model, risk-neutral agents arrive randomly at the market for an asset that has both common and private components to its value. Agents have different information about the common value (i.e., the present value of the future cash flows on the asset). Each agent chooses either to buy or sell one share. If his order does not execute, he revisits the market and can revise his order. A trader may reenter the market an arbitrarily large number of times before execution. Thus, agents face a dynamic problem: the actions they take at any point in time incorporate the possibility of future reentries. In addition, an agent may face adverse selection: prior to his first entry into the market, each agent chooses whether to buy information about the fundamental value of the asset. An informed agent views the current expected value of the cash flows on each entry, whereas an uninformed agent forms an estimate of this value based on market observables.

A model that incorporates the relevant frictions of limit order markets (such as discrete prices, staggered trader arrivals, and asymmetric information) does not readily admit a closed-form solution. As a result, we use numerical methods to solve for equilibrium. Once the algorithm has converged, we then simulate trader arrivals and analyze the results to determine the properties of market outcomes. Our paper is methodologically related to Goettler, Parlour, and Rajan (2005) but examines a different set of questions, and therefore presents a richer model. Most importantly, in this paper we model asymmetric information about the fundamental value of the asset. Thus, we can analyze the relationship between the volatility of changes in the fundamental value, the degree of asymmetric information, and transaction prices.

In our model, all traders are risk-neutral. Thus, the volatility of the fundamental value matters because it affects the value of the option provided to other agents by a limit order submitter. As a result, in equilibrium, this volatility affects liquidity provision, and hence, the overall informativeness of market observables such as transaction prices and order depths.

Overall, our findings include:

- Agents with no intrinsic motive for trade (i.e., speculators) are willing to pay the most for information, and also submit the bulk of limit orders to the market. Competition among speculators results in private information often being reflected in the limit order book.
- However, speculators supply less liquidity when the asset is more volatile. Instead, they opportunistically exploit their information and place market orders. Therefore, in high volatility assets, recent transaction prices are more informative (compared to bid and ask quotes) about the true value of the asset, and thus about future prices, than in low volatility assets.
- Depth in the limit order book is also informative about the true value of the asset. However, depth at and away from the quotes has different effects. Thus, selling pressure (depth at the ask) tends to lead to lower prices, whereas dispersion on the sell side (depth away from the ask) tends to lead to higher prices.

- There is a “volatility multiplier” effect: assets with high fundamental volatility also exhibit greater volatility in the microstructure noise (i.e., the deviation of transaction price from estimated fundamental value). In an ideal frictionless market, all trades should occur at the fundamental value, and the microstructure noise should be identically zero. Thus, the volatility of the microstructure noise is a measure of the level of trading frictions in the market. The effect is exacerbated when only speculators are informed, since quotes are more often set by uninformed traders.

More broadly, the set of agents posting the best limit prices (on either the buy or sell side) changes across time, leading to transaction prices that depart from the fundamental value of the asset. In our model, agents' responses to market frictions naturally create a time variance in transaction prices. This suggests that one explanation for time-varying expected returns or betas may be changes in the composition of the types of agents wishing to trade at any particular point of time.

In our simulations, transaction prices do not respond instantaneously to changes in the fundamental value, and the degree of inertia depends on both the volatility in the fundamental value and the proportion of agents who acquire information. The inertia, in turn, implies that the microstructure noise displays positive autocorrelation, and is negatively correlated with changes in the fundamental value. Further, changes in the microstructure noise are negatively correlated with changes in the estimated fundamental value, with the degree of correlation varying with the asset volatility, and the extent of asymmetric information.

These properties are important to account for in decomposing the transaction price into the efficient price (i.e., the fundamental value) and microstructure noise. Further, the negative correlation between changes in the microstructure noise and changes in the fundamental value implies that betas estimated in an asset pricing regression will be too low. This opens up the possibility that microstructure effects (including proxies for liquidity or idiosyncratic risk) may spuriously attain explanatory power in cross-sectional asset pricing regressions.

Our theoretical predictions are consistent with recent work that uses high-frequency stock market data. For example, Hansen and Lunde (2006) consider the stocks in the Dow Jones Index, and find that in many cases the microstructure noise is negatively correlated with the fundamental value. Ait-Sahalia, Mykland, and Zhang (2006) show that, controlling for trade reversals, the remaining component of microstructure noise displays positive autocorrelation for stocks such as Microsoft and Intel. Bandi, Moise, and Russell (2006) show that innovations in microstructure volatility may be a priced factor.

To estimate microstructure noise, Engle and Sun (2007) use reduced form statistical models. Diebold and Strasser (2007) base their estimates on the insights from static microstructure models. Such models have no role for the history contained in a limit order book, and its use in determining the expected value of the asset. In contrast,

our model is dynamic and shows the informativeness of the limit order book in predicting future price changes.

An extensive literature explores endogenous information acquisition in a rational expectations framework. Hirshleifer (1971) observes that, in an exchange economy with risk-averse traders, information has no social value: if all agents are informed, risk-sharing opportunities are eliminated and the market breaks down.<sup>1</sup> Grossman and Stiglitz (1980) note that if costly information is immediately impounded into price, agents should not acquire it. Clearly, the argument depends on how agents profit from their information, so the results are specific to a price formation mechanism.<sup>2</sup> Thus, a model with endogenous information acquisition should include stylized representations of the most important trading frictions. Previous general equilibrium work with endogenous information acquisition considers noise in the aggregate asset supply (e.g., Verrecchia, 1982; Admati and Pfleiderer, 1987) or “noise” traders with exogenous demands (Barlevy and Veronesi, 2000) to ensure that prices are only partially revealing. Our market is inherently dynamic, with the common value of the asset changing over time. In a temporal sense, informed traders are local monopolists. Hence, there can at best be partial revelation.

Risk-averse noise traders are considered in a Kyle (1985) framework by Spiegel and Subrahmanyam (1992), who demonstrate that these traders reduce the amount they trade in the presence of adverse selection. Mendelson and Tunca (2004) consider an informed insider who takes into account the effect of his acquiring information on the orders of risk-averse noise traders. Since market prices are partially revealing, the gains to trade are reduced when the insider acquires information. Thus, even at a zero cost, the insider may choose to not acquire information. Since traders are risk-neutral in our model, neither of these effects is present.

When there are multiple informed traders in a Kyle-type model, Foster and Viswanathan (1996) show that the correlation between informed traders’ signals is important in determining the speed of information revelation. Imperfect correlation eventually leads to a waiting game, so the equilibrium is characterized by less trade in later periods. As a result, the market may become illiquid towards the end of the overall trading period. If the initial correlation of traders’ signals is low enough, less information may be revealed than with a monopolist informed trader. Their numerical results are confirmed by Back, Cao, and Willard (2000) in a continuous time model. In our model, informed traders know the common value on each entry into the market. However, the common value changes over time. Hence, though signals are imperfectly correlated, agents have an incentive to act on information

before it becomes stale due to an exogenous change in the common value, and the market remains active.

Early work on the endogenous choice of limit versus market orders by informed traders includes Chakravarty and Holden (1995), who show that an optimal mix of limit and market orders leads to a higher payoff than submitting market orders alone, when there is uncertainty about the price at which market orders will execute. Kumar and Seppi (1994) also have informed investors submitting both limit and market orders, to avoid being detected in the midst of uninformed traders who are also employing a mix. Kaniel and Liu (2006) show that informed traders will use limit orders when information is sufficiently persistent. In our model, information is short-lived, but informed traders nevertheless submit a large proportion of limit orders. If they did not, uninformed traders would be faced with severe adverse selection, and would set wide spreads.

Our paper builds on the recent literature on dynamic limit order markets with strategic traders.<sup>3</sup> For example, Rosu (2007) presents a continuous time private value model of a limit order market with continuous prices and instantaneous punishment strategies. Foucault, Kadan, and Kandel (2005) characterize equilibrium in a dynamic limit order book with private values and differences in time preferences. None of these models consider information acquisition, or allow agents to differ in what they observe upon entering (or reentering) the market. Back and Baruch (2007) consider a continuous time model with asymmetric information, and demonstrate that, in the absence of frictions, market design is irrelevant (every equilibrium in a limit order market can be sustained as an equilibrium on a floor exchange with competitive market-makers, and vice versa). Our paper is complementary, in that we explicitly model frictions in a limit order market.

We outline our model in Section 2. Details of the model and numerical algorithm appear in the Appendix. In Section 3, we consider the value of information to different types of traders, and exhibit equilibria with endogenous information acquisition. Next, we examine how trader behavior and the resulting quotes differ with asymmetric information and fundamental properties of the asset in Section 4. We consider the price distortions introduced by the limit order market, and the asset pricing implications of these distortions, in Section 5. Section 6 concludes.

## 2. Model

We consider a dynamic model of trade in a single financial asset. Before participating in the market, traders decide if they want to become informed. After making their information acquisition decision, agents trade. In the market, time is continuous, and traders arrive randomly. A trader has one share to trade and may choose to buy or sell the asset. Traders also choose the price at which they

<sup>1</sup> Hakansson, Kunkel, and Ohlson (1982) demonstrate that information can have social value if the market is not allocationally efficient. Bernardo and Judd (1997) find that information acquisition reduces welfare both because uncertainty is resolved before trade (the Hirshleifer effect) and because rent-seeking trades by informed agents reduce optimal risk-sharing.

<sup>2</sup> For example, Jackson (1991) demonstrates that the price-taking assumption is critical in order to sustain the Grossman-Stiglitz paradox.

<sup>3</sup> Early work includes Parlour (1998), who characterizes a limit order market with no common value, and Foucault (1999), who models a common value, but truncates the book to one share.

place their order. If the order is not executed, the trader randomly returns to the market and can cancel and resubmit his order. Any unexecuted trader may reenter the market, which potentially allows a trader to repeatedly revisit the market before execution. The reentry and endogenous cancellation represent a significant modeling improvement over Goettler, Parlour, and Rajan (2005), who treat cancellation as exogenous, so that each trader essentially enters the market only once.

We turn to each of the model elements in more detail.

The limit order book is characterized by a discrete set of prices,  $\mathcal{P} = \{p^i\}_{i=-\infty}^{\infty}$ , at which traders may submit orders. The tick size or distance between any two consecutive prices is normalized to one. Associated with each price  $p^i \in \mathcal{P}$  at time  $t$  is a backlog of outstanding orders to buy or sell the asset,  $\ell_t^i$ . This backlog is the depth at price  $p^i$ . We adopt the convention that a positive quantity denotes buy orders and a negative quantity sell orders. The limit order book at time  $t$ ,  $L_t = \{\ell_t^i\}_{i=-\infty}^{\infty}$ , is the vector of outstanding orders.

Given a limit order book  $L$ , the bid price or quote is  $B(L) = \max\{i | \ell^i > 0\}$ , the highest price at which there is a limit buy order on the book, and the ask price or quote is  $A(L) = \min\{i | \ell^i < 0\}$ , the lowest price at which there is a limit sell order on the book. If the corresponding set of prices is empty, define  $B(L) = -\infty$  and  $A(L) = \infty$ .

Limit orders are executed according to time and price priority: that is, orders submitted earlier are further ahead in the queue. Buy orders at higher prices and sell orders at lower ones are accorded priority. Therefore, an order executes if no other orders have priority, and a trader arrives who is willing to be a counterparty. Further, limit orders are marketable; an order to buy at a price above the ask executes instantaneously at the ask, and is called a market order in what follows.

*Traders:* New traders arrive at the market according to a Poisson process with parameter  $\lambda$ . Each trader is allowed to trade at most one share of the asset: however, he may choose to buy or sell that share. On any entry, a trader may choose to submit an order for one share, or to not submit any order.

A trader who previously entered the market, but who has not yet traded a share, reenters the market at a random time determined by a Poisson process with parameter  $\lambda_r$ . Reentry, therefore, is not instantaneous, and represents a friction agents must take into account when submitting an order. The reentry friction captures the idea that agents do not continuously monitor the market. Traders are potentially active until their order executes, at which time they leave the market forever. Thus, at any point of time, there will be a random number of agents who have not yet traded. Each of these agents has the opportunity to reenter the market, even if they have done so previously. Thus, a trader may have several market visits before finally executing.

Upon reentry, a trader may leave an existing order on the book or cancel it and submit a new order. The benefit of retaining the existing order is that he maintains his time priority (his place in the queue). The cost is that the asset value may have moved in a manner that affects the expected payoff from the order. For example, if he

submitted a buy order and the asset value has since fallen, his order may now be priced too high. Conversely, if the asset value has since risen, his order may be at too low a price, and there may be little chance of it executing. Further, a trader may also find that the priority of a previous order has changed by the time he reenters the market.

*Information:* At any instant  $t$  the asset has a common or fundamental value, denoted  $v_t$ . The fundamental value is the expectation of the present value of future cash flows on the stock, and evolves as a random walk. Innovations in the fundamental value occur according to a Poisson process with parameter  $\mu$ . If an innovation occurs, the fundamental value increases or decreases by  $k$  ticks, each with probability  $\frac{1}{2}$ . Changes in the fundamental value reflect new information about the firm or the economy. On his first entry to the market, an agent may choose to buy information by paying a cost  $c \geq 0$ . Incurring this cost gives an agent access to a service that reports the current value of  $v$  on this and each subsequent entry. Since all investors have a chance to acquire the information, it is publicly available: for example, information reported in financial statements, Securities and Exchange Commission (SEC) filings, or analyst reports, or prices of related assets such as options.

Uninformed agents view  $v$  with a time lag,  $\Delta_t$ , measured in units of time. That is, an uninformed agent in the market at time  $t$  knows  $v_{t-\Delta_t}$ , whereas an informed agent in the market at time  $t$  knows the current value  $v_t$ . What is important for our results is that one group of agents has a better estimate of the value of the asset. Modeling the uninformed as those observing  $v$  with a lag is a tractable way of doing this.

In addition, all agents observe the history of the game. Let  $t$  denote a time at which an agent has entered the market, and let  $h_t$  denote the history up to time  $t$ , before the agent takes an action. The history includes all actions in the game until time  $t$  as well as changes in the fundamental value until time  $t$  (for informed traders) and time  $t - \Delta_t$  (for uninformed traders). For all agents, the limit order book  $L_t$  provides information about current trading opportunities. Uninformed agents use their information set to update their expectation about the fundamental value  $v$ . The history offers strategic information to informed agents as well: using the information available to an uninformed agent allows informed agents to better predict the actions of uninformed agents, and thus earn a higher payoff themselves.

*Traders' optimization:* Each trader has a type  $\theta = \{\rho, \alpha\}$ , where  $\rho$  is a discount rate and  $\alpha$  a private value for the asset. The payoff a trader earns as a result of trading is discounted back to his first arrival time in the market at the rate  $\rho$ . The cost of delaying trade could include an opportunity cost (e.g., if a trader is executing a trading strategy across different assets and must delay trades in other assets) and a cost to monitoring the market before execution, rather than the time value of money.<sup>4</sup>

<sup>4</sup> Traders in some financial markets appear to care about differences in seconds in the time to execution; the discount rate captures this desire to trade early.

The private value  $\alpha$  represents private benefits of trade, accruing to a trader as a result of liquidity shocks or private hedging needs, and is independently drawn across traders. Let  $F_\alpha$  denote the distribution of  $\alpha$ . Traders are risk-neutral and submit orders to maximize their expected discounted payoff. Utility is earned only if an order executes. For a particular trader  $\theta = \{\rho, \alpha\}$ , the instantaneous utility at time  $t$  is

$$u_t = \begin{cases} \alpha + v_t - p^i & \text{if he executes a buy order at price } p^i \text{ and time } t, \\ p^i - \alpha - v_t & \text{if he executes a sell order at price } p^i \text{ and time } t, \\ 0 & \text{if he does not execute an order at time } t. \end{cases} \quad (1)$$

The expected payoffs to different actions depend on a trader's information set. For each agent, this includes his private valuation for the asset,  $\alpha$  and information about  $v_t$ .

*Equilibrium:* To find the equilibrium of the model, we first fix information acquisition strategies for each type of trader, and then solve for optimal trader strategies in the resulting trading game. The payoffs in the trading game then allow us to determine the information costs which sustain the conjectured information acquisition strategies.

In the trading game, each time a trader is in the market, he chooses an action that maximizes his expected discounted utility, given the state he observes. Optimal strategies in this model, therefore, are state-dependent. If a state is defined to be the history of events in the game observed by the trader, his decision is Markovian.

Formally, the trading game is a Bayesian game. Traders have privately known utilities from trade (since a trader's  $\alpha$  is unknown to other traders) and possibly private information about the fundamental value  $v$ . As [Maskin and Tirole \(2001\)](#) point out, the proper solution concept here is Markov perfect Bayesian equilibrium, which requires traders to play dynamically optimal strategies on each entry into the market, given their current beliefs. We focus on stationary, symmetric equilibria, in which each type of trader chooses the same strategy, and this strategy does not depend on the time at which the trader arrives at the market.

*Solving for equilibrium:* Since an analytic solution is not feasible, we numerically solve for equilibrium in the trading game, using a natural extension of the simulation-based algorithm of [Pakes and McGuire \(2001\)](#) for complete information games. A transparent difference is that different agents have different state variables (since some agents may know the fundamental value  $v$ , whereas others do not). In this algorithm, traders start with beliefs about payoffs to different actions, and update these beliefs when they take an action and observe its realized payoff. A key step in updating their beliefs over different actions is determining the expected fundamental value of the asset when an agent is uninformed.

In principle, the state for a trader includes the entire history of the game. However, in order to make the problem computationally tractable, we need to impose specific restrictions on the state space.<sup>5</sup> In the Appendix,

we provide a detailed mathematical description of the trader's decision problem, the state space used in the simulations, the algorithm used to obtain a numerical solution of the trading game, and the convergence criteria. To ensure that traders learn the payoffs to all available actions, we require them to tremble (i.e., choose actions that are suboptimal given their current beliefs) with small probability while the algorithm is converging.

## 2.1. Numerical parameterization of the trading game

For our numeric simulations, we use parameters that have been identified by the existing empirical literature.

- We normalize the mean time between new trader arrivals,  $\lambda$ , to one. A unit of time in our numerical solution thus represents the average time between new trader arrivals. In any stationary equilibrium, trades must happen on average every two periods. If a period is assumed to be one minute long, approximately 180 trades would occur in the course of a normal trading day.
- On average, a trader reenters the market after four units of time. Reentries are independent across traders and entries. Numerically, it is straightforward in the algorithm to consider reentry rates that differ across types and trader information. In this paper, however, we are primarily interested in isolating the effects of differential information on market outcomes, so we keep the reentry rate the same across informed and uninformed traders. Conceptually, we think of reentry rates as depending on the cost of monitoring the market, with a lower monitoring cost implying a higher rate of reentry.
- The support of the discrete  $\alpha$  distribution in ticks is  $\{-8, -4, 0, 4, 8\}$ . The probabilities are 15% each on  $-8$  and  $8$ , 20% each on  $-4$  and  $4$ , and 30% on zero. A tick in the simulation corresponds to one-eighth of a dollar. The traders with  $\alpha = 0$  constitute traders who may be willing to buy or sell, depending on the state of the market when they arrive. We refer to these agents as "speculators," since they have no intrinsic motive to trade. The traders with  $\alpha \in \{4, 8\}$  are likely to be buyers, and those with  $\alpha \in \{-4, -8\}$  are likely to be sellers. These characterizations are borne out in our simulations. Our private value distribution approximately corresponds to the findings of [Hollifield, Miller, Sandás, and Slive \(2006\)](#), who estimate the distributions of private values for three stocks on the Vancouver exchange. Since they consider a world with symmetric information, their estimate of the gains to trade likely represents an upper bound (since some trading will occur for informational reasons). Our parameterization of  $F_\alpha$  is based on their identification of three types of traders within these distributions. They find that, on average across the three stocks, 44% of traders have private values within 2.5% of the fundamental value of the stock, 26% have values that differ from the

<sup>5</sup> In particular, these restrictions exclude some potentially payoff-relevant variables, such as the exact time at which different events happened.

fundamental value by 2.5% to 5%, and 30% have values that differ from the fundamental value by more than 5%. This corresponds approximately to the probabilities of our three kinds of traders. In terms of ticks, on average across the three stocks, 2.5% of the fundamental value translates to approximately 3.45 ticks, and 5% of the fundamental value to approximately 6.9 ticks.

- Changes in  $v$ , the fundamental value of the asset, occur at times drawn from a Poisson distribution with the expected time between changes being eight units of time. We consider two sets of models:
  - (i) Low volatility: whenever  $v$  changes, it increases or decreases by one tick, each with probability  $\frac{1}{2}$ .
  - (ii) High volatility: whenever  $v$  changes, it increases or decreases by two ticks, each with probability  $\frac{1}{2}$ .

Our low volatility parameterization also roughly corresponds to the findings of Hollifield, Miller, Sandás, and Slive (2006). For the three stocks they consider, they report the volatility of the midpoint of the bid and ask quotes over 10-minute intervals, to be approximately 1.70%. Using a stock price of \$15, that translates to a variance of \$0.065025 (or six cents) over a 10-minute interval. With a tick size of one-eighth, the variance in ticks over a 10-minute interval is 4.16 ticks. The midpoint of the bid and ask quotes is a rough proxy for the fundamental value. Assume that two units of time in our model correspond to one minute of real time. Since  $\lambda$  denotes the expected number of innovations to the fundamental value in one unit of model time, over a 10-minute period we have on average  $20\lambda$  innovations. In the low volatility case, since each innovation corresponds to one tick, the expected sum of squares of changes in the fundamental value is also  $20\lambda$  over a 10-minute interval. Using  $\lambda = 0.125$  (so that an innovation occurs every eight units of model time on average), the expected sum of squares of changes in the fundamental value over a 10-minute interval in our model is 2.5 (in ticks) in the low volatility case, and 10.0 in the high volatility case. Since our innovation process has a zero mean, the expected sum of squares is comparable to the variance reported by Hollifield, Miller, Sandás, and Slive (2006).

- We set  $\Delta_t$ , the lag with which an uninformed trader observes  $v$ , to be 16 units of time. We varied  $\Delta_t$  in our simulations, increasing it to a maximum value of 128, and found no significant differences in the results for larger values.
- $\rho$ , the continuous discount rate, is the same for all agents and is set to 0.05. Recall that this is not the time value of money, but rather a preference parameter that reflects the cost of not executing immediately.<sup>6</sup>
- Limit orders may be submitted at any feasible price that lies in a range between 6.5 ticks above and below an agent's expectation of  $v$ . For an informed trader, this

expectation is just the current value of  $v$ . For traders who observe  $v$  with a lag  $\Delta_t$ , this expectation is their best estimate given the lagged fundamental value, the current book, and the observed market history.<sup>7</sup> Market orders, of course, may be submitted at the current bid (market sells) or ask (market buys) regardless of an agent's expectation of  $v$ .

### 3. Information acquisition and trading behavior

Traders only acquire costly information if they can use it to sufficiently increase their expected payoff in the trading game. Rational agents who do not acquire information, however, anticipate that informed agents may be present in the market, and therefore adopt trading strategies that account for this fact. We start by analyzing the gains to acquiring information, given that all traders know other agents' information acquisition strategies.

#### 3.1. Information acquisition

To determine each trader type's willingness to pay for information, we fix agents' information acquisition strategies and solve for equilibrium in the trading game. Thus, we take into account how agents' strategic behavior changes in response to adverse selection. We consider symmetric equilibria in which all agents with a given  $\alpha$  (i.e., private value) take the same action at the information acquisition stage. We then consider the payoff to a particular type who deviates.

Radner and Stiglitz (1984) demonstrate that information is valuable to a single Bayesian decision-maker only if it induces a change in her action. In our model, speculators are the agents most likely to change their action (i.e., switch from buying to selling the asset, or vice versa) based on information about the value of the asset. Indeed, we find that speculators have the highest willingness to pay for information. Verrecchia (1982) shows a similar result in a general equilibrium rational expectations framework—the least risk-averse agents (i.e., those with the lowest intrinsic motive to trade) acquire costly information.

**Observation 1.** *Agents' willingness to pay for information decreases in the absolute value of  $\alpha$ .*

To reach Observation 1, we examine different information regimes. In each regime, we first fix the information acquisition strategy of each trader type. Once the algorithm has converged, we hold agents' beliefs fixed, set the tremble probability to zero, and simulate a further 300 million market entries (including reentries by returning traders). The outcomes obtained by traders in the new

<sup>6</sup> We experimented with lower values of  $\rho$ , and found the results to be qualitatively similar. However, traders took longer to execute on average, and the recursive set of states (which determines the algorithm's speed) was considerably larger.

<sup>7</sup> We simulated versions of the models in which limit orders could be submitted further away from the fundamental value. Although orders were occasionally submitted at such ticks, these orders rarely executed, appearing to substitute for not submitting an order at all. There was no appreciable effect on market outcomes, either in the aggregate or for any particular type of trader.

**Table 1**

Average gross payoff in ticks to each type of trader, across different information and volatility regimes.

This table shows the average payoff in ticks to each type of trader in different information and volatility regimes. The type of a trader is indicated by the value of  $\alpha$ . Since the model is symmetric, positive and negative  $\alpha$  values with the same absolute value are combined for reporting purposes. The row labeled “Equilibrium” shows the payoff to each trader type in equilibrium. The row labeled “Deviation” shows the payoff if that type of trader were to deviate in information acquisition. The averages are determined as mean payoffs over 300 million market entries (including new and returning traders). Standard errors on the mean payoffs are less than 0.0005 for equilibrium strategies and less than 0.0020 for deviator strategies. Payoffs in *italics* indicate informed agents.

Information regime		Low volatility			High volatility		
		Value of $ \alpha $			Value of $ \alpha $		
		0	4	8	0	4	8
All agents informed	Equilibrium	0.438	3.511	7.319	0.555	3.452	7.180
	Deviation	0.230	3.424	7.258	0.321	3.301	7.057
	Value of information	0.208	0.087	0.061	0.234	0.151	0.123
Speculators informed	Equilibrium	0.675	3.454	7.177	0.883	3.181	6.871
	Deviation	0.432	3.514	7.190	0.406	3.458	7.083
	Value of information	0.245	0.060	0.013	0.477	0.277	0.212

simulation are averaged to determine the expected consumer surplus (i.e., the equilibrium payoff or expected utility) for each type.<sup>8</sup>

Next, we allow a small mass of each type (2%) to deviate in information acquisition and then trade optimally. We ensure that at most one deviator is present in the market at any given time, to preserve the spirit of unilateral deviation. Non-deviator agents in the simulation play the same strategies as in the equilibrium of the original trading game. The strategies and payoffs for the deviators (and only the deviators) are determined afresh by the algorithm. Again, once the algorithm has converged, we hold the deviators' beliefs fixed and simulate a further 300 million trader entries, to determine expected payoffs to the deviators.

The gross payoff (i.e., ignoring the cost of acquiring information) of each type in each of two information regimes (all agents acquiring information, and only speculators acquiring information) is reported in Table 1, for both the low and high volatility cases.<sup>9</sup> All payoffs are quoted in ticks. The value of information to each type of agent is represented by the difference in payoffs between being informed and remaining uninformed.

As seen from the Table 1, in both volatility regimes, the value of information decreases in the absolute value of  $\alpha$ . Information is most valuable to speculators (i.e., agents with  $\alpha = 0$ ), who have no intrinsic benefit to trade. These agents are willing to take either side of the market, depending on the available payoff. Conversely, agents with  $\alpha = 8$  are unlikely to switch from buyers to sellers, so

information is less valuable to these agents. As expected, when volatility is high, information is more valuable to each trader type.

Endogenous information acquisition equilibria can be gleaned from Table 1 as well. For example, for any information cost between zero and 0.061 ticks, there is an equilibrium in which all agents purchase information, even when the asset has low volatility. Similarly, if the asset has high volatility, and the information acquisition cost is between 0.277 and 0.477 ticks, in equilibrium only speculators are informed.

As may be seen from the table, multiple equilibria may exist in the information acquisition game. For example, if the asset has low volatility and the information acquisition cost is 0.0605, an equilibrium is for all agents to purchase information. However, another equilibrium is for only speculators to purchase information.

If a larger proportion of agents acquire information, the probability an uninformed agent will trade with an informed one is higher. However, when more agents are informed, uninformed agents can obtain a more precise estimate of the fundamental value from observing market prices and transactions. The overall value of information to an agent depends on the tradeoff between these two effects, and, for some cost values, the information acquisition game has the flavor of a coordination game.

The broad insight that emerges is that in any information regime, speculators have the greatest willingness to pay for information. Thus, for the rest of the paper, we suppress discussion of endogenous information acquisition, and focus on the two information regimes shown in Table 1: either all agents acquire information or only speculators acquire information. In conjunction with the two volatility regimes (high and low), we thus have a total of four regimes to focus on.

<sup>8</sup> We use a large number of simulated traders so that standard errors of measures of interest are sufficiently small.

<sup>9</sup> We examined all feasible information regimes. The value of information remains monotonic in the absolute value of  $\alpha$  in other regimes as well. For brevity, other regimes are not reported here.

**Table 2**

Proportion of time, in percent, that each trader type has a limit sell at and away from the ask price.

This table shows the percentage of time that each type of trader has a limit sell at or above the ask price. We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). These one million entries form the data on which this table is based. The average proportion of time for each trader type is found as follows. We observe the market every 10 minutes. At each observation, each trader type is assigned a one if at least one trader of that type has an order at the ask, and zero otherwise. We compute the average proportion of time that each trader type has limit sells above the ask in a similar fashion. The reported average for each type is a mean across all such observations.

Volatility regime	Information regime	At or above ask	Value of $\alpha$					
			–8	–4	0	4	8	
Low	All agents informed	At ask	7.4	28.1	59.5	0.1	0.0	
		Above ask	1.0	5.0	50.0	0.0	0.0	
	Speculators informed	At ask	6.8	30.0	57.0	0.2	0.0	
		Above ask	2.2	9.8	53.8	0.3	0.0	
	High	All agents informed	At ask	7.2	32.8	47.9	0.2	0.0
			Above ask	1.6	10.9	57.3	0.0	0.0
Speculators informed		At ask	11.6	35.5	46.3	0.0	0.0	
		Above ask	3.2	15.8	65.6	0.0	0.0	

For all remaining results in the paper, we find the equilibrium under each regime, hold beliefs fixed, disallow trembles, and simulate one million trader entries (including reentries). All averages are therefore based on a large sample. Standard errors on reported means are on the order of the third decimal place, and are hence not reported.

#### 4. Trading behavior and learning

How do informed traders optimally act on their information? Specifically do they submit market orders or limit orders, and if so at what prices? Their behavior determines the information content of the limit order book. Recall that in a dynamic limit order market the best bid and ask quotes come from previously submitted limit orders that have not executed. If quotes are set by informed agents, the bid and ask prices will reflect private information. By contrast, if informed traders submit market orders then transactions reflect current mispricing and should be followed by price changes.

In what follows, we report the average behavior of particular trader types. Specifically, we average across different states that such an agent faces on entering in the market. When we compare these behaviors across different regimes, both the chances of particular states differ and agents' optimal strategies differ.

##### 4.1. Liquidity supply

Supplying liquidity in this market requires not just posting a limit order, but doing so at a competitive price. Thus, to examine liquidity provision, we consider the proportion of time that each trader type has a limit sell at the ask price and at a price higher than the ask. That is, we sample the limit order book at intervals of 10 units of time. At each point of time, for each order on the book, we

determine the private value of the agent who submitted that order. The results are reported in Table 2.<sup>10</sup> Since our model is symmetric, the data on limit buys at the bid price are similar.

**Observation 2.** (i) *Quotes are on average set by speculators, and an even greater proportion of orders away from the quotes come from speculators.*

(ii) *When there is high volatility, speculators reduce their provision of liquidity at the quotes, and agents with an intrinsic motive for trade increase liquidity provision. This effect is strongest when only speculators are informed.*

Speculators, the traders with no intrinsic motive for trade, tend to supply liquidity to the market. Because such traders are more likely to be informed, limit orders are more likely to be submitted by informed traders. That is, informed traders in our market tend to submit limit orders. However, this observation is different from the finding in the one-period experimental results of Bloomfield, O'Hara, and Saar (2005) that informed traders supply liquidity. In our model speculators have the highest demand for information and the lowest desire for trade, and therefore both acquire information and supply liquidity. However, when they are informed (and others are not), they somewhat decrease the amount of liquidity they supply.

This effect is manifested in two ways in Table 2. First, fixing the volatility regime, speculators are slightly less likely to have orders at the ask when only they are informed. While this effect is small, speculators are significantly more likely to have orders away from the quotes when only they are informed. Further, as we show in Table 5, in the latter case, an informed trader frequently uses market orders to take advantage of stale limit orders

<sup>10</sup> Note that the proportions across rows in Table 2 sum to less than 100, since sometimes the book is empty on that side of the market.

**Table 3**

Average ask quote (in ticks relative to the fundamental value  $\nu$ ) when each type of trader has an order at the ask price.

This table shows the average ask quote (in ticks above the fundamental value) conditional on a given type of trader having an order at the ask price. We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). These one million entries form the data on which this table is based. The average ask quote for each trader type is found as follows. We observe the market every 10 minutes and determine the average ask quote across all observations at which a given type of trader has an order at the ask price. The reported average for each type is a mean across all such observations. Agents with  $\alpha = 4$  and 8 are omitted from the table because they submit too few limit sells for the prices to be meaningful. The last column, "Average ask price," is an average of the ask price across all books in which the sell side is non-empty.

Volatility regime	Information regime	Value of $\alpha$			Average ask price
		–8	–4	0	
Low	All agents informed	–0.31	0.10	0.97	0.63
	Speculators informed	–0.41	0.18	1.13	0.73
High	All agents informed	–0.38	0.49	2.38	1.48
	Speculators informed	0.21	0.75	2.49	1.56

**Table 4**

Average price at which traders submit limit sells in ticks, relative to their own expectation of  $\nu$ .

This table shows the average price at which each type of trader submits a limit sell order. The price is shown in ticks relative to the trader's expectation of the fundamental value  $\nu$ . We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). These one million entries form the data on which this table is based. For each limit order to sell that is submitted, we find the price in ticks relative to the trader's expectation of  $\nu$ . The reported average for each trader type is a mean price across all limit orders to sell submitted by that type. Agents with  $\alpha = 4$  and 8 are omitted from the table because they submit too few limit sells for the prices to be meaningful.

Volatility regime	Information regime	Value of $\alpha$		
		–8	–4	0
Low	All agents informed	–0.45	0.01	1.46
	Speculators informed	–0.47	0.26	1.66
High	All agents informed	–0.66	0.24	3.79
	Speculators informed	–0.26	0.75	3.98

and so decreases his liquidity provision. This observation is also contrary to the observations of Kaniel and Liu (2006) who find that if information is sufficiently long-lived then limit orders might be preferable to informed agents.

A trader who provides a limit order at the quotes represents a potential marginal trader in the market. If a newly arriving agent submits a market buy (sell) order, the order will execute at the ask (bid) quote. Suppose all else is equal, and consider two assets with different fundamental volatilities. On average across time, these assets will exhibit different distributions of marginal traders, and hence different "representative agents." This effect obtains despite all agents being risk-neutral (so wealth effects are absent), and is a result of traders changing their order submission strategies across different regimes. Hence, one implication of Observation 2 is that any attempt at aggregation across traders will lead to results that vary by market or asset.<sup>11</sup>

#### 4.2. Aggressiveness of quotes

How close are quotes to the fundamental value of the asset? Consider the ask quotes. Since speculators have zero intrinsic motive for trade, on average, they should demand a price higher than the fundamental value when they sell the asset. Conversely, agents with a private value of  $-8$  should be willing to sell the asset at lower prices, sometimes even lower than the fundamental value. Thus, comparing across volatility regimes, the shift in the marginal trader toward lower private values as volatility increases leads to a reduction in the average ask price, bringing quotes closer to the fundamental value.

However, as we show in Table 3, this effect is dominated by a change in the order submission strategy of each type of agent. When volatility is high, speculators submit more conservative orders. Specifically, when speculators have orders at the ask price, the ask quote is about 1.4 ticks higher in the high volatility regimes. While agents with an  $\alpha$  of  $-4$  also submit more conservative orders when there is high volatility, those with the most extreme private values ( $\alpha = -8$ ) tend to be a little more aggressive when all agents are informed, but significantly more conservative when only speculators are informed. These agents are keen to trade quickly, and the trade-off between submitting more aggressive orders which

<sup>11</sup> In related work, Rindi (2008) demonstrates the ambiguous effect of market transparency on traders' incentives to acquire information and supply liquidity in a Walrasian market, while Boulatov and George (2008) consider strategic liquidity provision by informed agents.

execute more quickly and more conservative ones that offer better terms of trade matters more to them than to the types with more moderate private values.

The overall effect, as shown in Table 3, is to increase the average ask quote in the high volatility regimes, compared to the low volatility ones. The results on the average bid quote are exactly similar. Thus, bid–ask spreads are wider when volatility is high.

**Observation 3.** *In the high volatility regimes, speculators raise their ask quotes by more than the other trader types.*

Since speculators have a zero intrinsic value to trade, they must increase their ask prices substantially in the high volatility regime in order to provide a cushion against adverse movements in the fundamental value. In contrast, traders with extreme values (say,  $-8$ ) face a genuine tradeoff: increasing the ask price does provide an added cushion, but reducing the ask price may lead to quicker trade, and thus also reduce adverse selection possibilities. As Table 3 shows, the average ask is actually lower under high volatility (compared to low volatility) when all agents are informed and traders with  $\alpha = -8$  have an order at the ask.

Observation 3 is directly supported by examining the prices at which different types of traders submit limit orders. As Table 4 shows, in moving from low to high volatility, speculators substantially increase the price at which they are willing to sell the asset, by about two- and

one-quarter ticks in each information regime. Agents with middle values ( $\alpha = -4$ ) also raise their ask prices under high volatility, but by much less. Agents with extreme values ( $\alpha = -8$ ) actually lower their ask prices under high volatility when all agents are informed, in a desire to trade quickly.

#### 4.3. Liquidity demand, or transactions

Traders who submit market orders demand liquidity in this market. As shown in Table 5, the demand for liquidity shifts across trader types when the fundamental volatility increases. In low volatility regimes, speculators submit relatively few market orders, preferring to trade via limit orders. They substantially increase their demand for liquidity in high volatility regimes, especially when no other traders are informed. Comparing the speculators informed case across low and high volatility regimes, the percentage of market orders submitted by speculators more than triples, from 11.8% to 37.1%. Therefore, in markets with high fundamental volatility, transactions are more likely to be informative of future price movements as speculators submit market orders.

There are two main reasons why speculators switch from supplying to demanding liquidity when volatility changes from low to high. First, limit orders become riskier and are more susceptible to being picked off. Lacking an intrinsic motive for trade, speculators are especially susceptible to this effect. Second, especially

**Table 5**

Percentage of market orders submitted by each trader type.

This table shows the proportion of market orders submitted by each trader type, as a percentage of all market orders. We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). These one million entries form the data on which this table is based. For each trader type, we find the percentage of market orders submitted.

Volatility regime	Information regime	Value of $\alpha$				
		-8	-4	0	4	8
Low	All agents informed	23.1	20.1	13.3	20.3	23.2
	Speculators informed	24.7	19.3	12.2	19.2	24.7
High	All agents informed	20.2	17.7	24.4	17.7	20.0
	Speculators informed	16.6	15.0	36.9	15.0	16.5

**Table 6**

Average of price (relative to fundamental value) in ticks, for all executed sell orders, market and limit.

This table shows the average price at which each type of trader submits a limit sell order. The price is shown in ticks relative to the fundamental value  $v$ . We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). These one million entries form the data on which this table is based. For each executed sell order, market and limit, we find the price in ticks relative to  $v$  at the time of execution. The reported average for each trader type is a mean price across all executed sell orders that were submitted by that type. Agents with  $\alpha = 4$  and  $8$  are omitted from the table because they submit too few sell orders for the prices to be meaningful.

Volatility regime	Information regime	Value of $\alpha$		
		-8	-4	0
Low	All agents informed	-0.50	-0.16	0.70
	Speculators informed	-0.68	-0.27	1.03
High	All agents informed	-0.61	-0.18	0.85
	Speculators informed	-0.78	-0.38	1.26

**Table 7**

Mean and standard deviation of errors in beliefs of uninformed agents, in ticks.

This table shows the mean and standard deviation of errors in beliefs of uninformed agents. We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). These one million entries form the data on which this table is based. For each entry or reentry to the market by any agent, informed or uninformed, we construct the expectation about  $v$  for an uninformed agent. The error in belief is defined as the expected value of  $v$  minus the true value of  $v$ , in ticks.

Volatility regime	Information regime	Mean (std. dev.) of errors in beliefs
Low	All agents informed	0.00 (0.61)
	Speculators informed	0.00 (1.01)
High	All agents informed	0.00 (0.71)
	Speculators informed	−0.01 (1.69)

when no other traders are informed, speculators are more likely to find mispriced orders in the limit order book when volatility is high, making market orders more profitable.

**Observation 4.** *Speculators substantially increase their demand for liquidity when the fundamental value has high volatility. The effect is heightened when no other trader types are informed.*

#### 4.4. Benefiting from trade

Traders choose their order type and the prices at which they trade in order to get the best possible terms of trade from other market participants. The best way to extract surplus depends both on asset characteristics and on others' optimal response to changes in asset characteristics. When there is high volatility, agents with a high absolute private value have a desire to trade quickly, even though they obtain worse prices at execution.

**Observation 5.** *An increase in volatility leads to speculators obtaining better terms of trade, whereas agents with a high intrinsic benefit to trade obtain worse terms of trade.*

In Table 6, we report the difference between transaction price and fundamental value for all executed sell orders (i.e., market and limit). An increase in this difference across information regimes signifies an improvement in the terms of trade for a given trader type.

Two comparisons are useful from the table. First, comparing across information regimes (i.e., comparing the first and second rows of numbers or the third and fourth rows), speculators experience an improvement in the terms of trade when only they are informed. When the volatility is low, sell orders of speculators execute 1.03 ticks above the fundamental value when only they are informed, as compared to 0.70 ticks above the fundamental value when all agents are informed.<sup>12</sup> The effects of adverse selection are exhibited by the increased cost (in terms of amount paid in excess of the fundamental value) paid by agents with nonzero private values ( $\alpha = -8$  or  $-4$ ) in the case in which only the speculators are informed.

<sup>12</sup> The notion that limit order submitters execute at favorable prices is consistent with the empirical work of Biais, Bisiere, and Spatt (2003), who fail to reject the hypothesis that competing limit order submitters on Island (an electronic limit order book) make positive profits.

Second, comparing across volatility regimes, speculators see a substantial improvement in their terms of trade when the fundamental volatility is high and they have superior information. For example, comparing the second row with the fourth row, speculators improve their terms of trade by 0.23 ticks per share when the volatility is high. Conversely, both asymmetric information and volatility lead to worse terms of trade for agents with nonzero private values (i.e.,  $\alpha = -8, -4$ ).

#### 4.5. Learning by uninformed traders

We next consider the beliefs of uninformed traders over the fundamental value of the asset. In regimes in which uninformed traders exist, these beliefs are essential to determining their expected payoffs from different actions. When all agents in the market are informed, we construct the expectation over fundamental value of a hypothetical uninformed agent (see footnote 20 in the Appendix for further details). With a slight abuse of terminology, we thus refer to the beliefs of an uninformed trader even when all traders are informed about the fundamental value.

Recall that an uninformed trader in the market at time  $t$  observes a lagged fundamental value  $v_{t-\Delta t}$ . He uses market observables (including the limit order book and information about the most recent transaction) to update his beliefs about the fundamental value.

In Table 7, we show the mean and standard deviation of the errors in beliefs of an uninformed trader, across the four regimes considered. The belief error is defined as the expected value of  $v_t$  for an uninformed agent, given market conditions at time  $t$ , minus the current value of  $v_t$ . As expected, the mean error is close to zero, and the standard deviation increases when there is high volatility and when only speculators are informed.

We next consider a simple linear OLS specification for belief formation in each of the two volatility regimes in which only speculators are informed, using data from the simulations. Since successive observations from the market will exhibit serial dependence, we consider observations that are separated from each other by 100 trader entries (including reentries).

The dependent variable in the regressions is the extent to which an uninformed agent uses market observables to revise her belief about the value of  $v_t$ . That is, the

**Table 8**

Regression of belief updates by uninformed traders when only speculators are informed.

This table reports coefficient estimates and  $t$ -statistics from an OLS regression of the expected value of  $v_t$  for an uninformed trader in the market at time  $t$ , relative to the last value observed by the trader,  $v_{t-A_t}$ , on various price and depth variables. The dependent variable is measured in ticks. Each price variable on the right-hand side is the relevant price minus the last observed value,  $v_{t-A_t}$ , also in ticks. Each depth variable is measured in shares. We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). For this table, we include every one hundredth entry or reentry in that sample of one million, keeping only those observations for which the book contains at least one limit order on each of the buy side and the sell side. The regimes considered are the ones in which only speculators are informed.

Independent variable	Volatility regime			
	Low		High	
	Coeff.	$t$ -stat.	Coeff.	$t$ -stat.
Constant	0.036	(2.46)	0.031	(1.32)
Ask price	0.333	(50.33)	0.261	(78.29)
Bid price	0.393	(57.87)	0.256	(78.05)
Last transaction price	0.081	(16.05)	0.365	(90.57)
Last transaction sign	-0.015	(-3.26)	0.106	(20.39)
Depth at ask	-0.068	(-19.95)	-0.130	(-9.70)
Depth at bid	0.070	(20.17)	0.096	(7.22)
Depth above ask	0.037	(11.82)	0.062	(20.83)
Depth below bid	-0.037	(-12.43)	-0.060	(-20.76)
No. of observations		8,420		8,721
$R^2$		0.880		0.961

dependent variable is  $E[v_t|s_t] - v_{t-A_t}$ , where  $s_t$  denotes the state observed by the agent in the market at time  $t$ . The dependent variable in each volatility regime has a mean that is not significantly different from zero.

The right-hand side variables include ask and bid quotes, the last transaction price, the sign of the last transaction (i.e., +1 if the transaction resulted from a market buy, and -1 if it resulted from a market sell), and the depths on either side at and away from the quotes. For all price variables, we use the price minus the last observed value of  $v$ . We restrict attention in each regime to books that are non-empty on both sides of the market. The results of the regressions are shown in Table 8.

As shown in the table, a linear approximation appears to be a good fit for the belief formation process of uninformed traders. If the ask and bid quotes each increase by a tick (or, conversely, the mid-point of the ask and bid quotes increases by a tick), the expected value of  $v_t$  increases by about 0.5–0.7 ticks, depending on the regime. Further,

**Observation 6.** (i) *Uninformed traders' beliefs place substantially greater weight on the last transaction price when there is high volatility.*

(ii) *Knowing that the previous transaction was a market buy has a greater positive impact on the expected fundamental value under high volatility.*

(iii) *Depth at the ask reduces expectations about  $v$ , and depth away from the ask increases expectations about  $v$ , with depth at and away from the bid having a symmetric effect.*

The intuition for (i) and (ii) in Table 8 is that informed traders (i.e., speculators) increase their proportion of market orders under high volatility. That is, they increase

their demand for liquidity, and somewhat reduce their supply. Hence, the price at which the previous transaction took place is substantially more informative (both economically and in terms of statistical significance) about the current value of  $v$  when the asset has high volatility. Further, uninformed traders typically submit market buys at a price higher than the fundamental value, whereas informed traders invariably submit market buys at prices lower than the fundamental value. On average, therefore, the price of a market buy is above the fundamental value when there is low volatility, and below the fundamental value when there is high volatility. This is reflected in uninformed traders' beliefs.<sup>13</sup>

Limit order depth is also informative about the current value. Notice that depth at and away from the quotes has opposite effects: increased depth at the ask suggests the current value is lower than expected, whereas increased depth away from the ask suggests that the current value is higher than expected. Recall that traders who submit limit orders are able to revisit the market and resubmit orders, and on average reenter the market twice as often as the fundamental value changes. Thus, there are few stale orders on the book. Orders submitted away from the ask suggest the current ask is too low, and hence lead to an upward revision in beliefs about  $v$ . On average, both transaction prices and traders' beliefs are "correct" (that is, average out to the true value). Thus, selling pressure (depth at the ask) tends to lead to lower prices, whereas

<sup>13</sup> The negative sign on the last transaction sign under low volatility in Table 8 is counter-intuitive, and is not robust to model specification. We also performed a two-stage procedure in which the last transaction price is first regressed on the last transaction sign, with the residual being used as the independent variable in the belief revision regression. This results in a positive coefficient on the last transaction sign under low volatility as well.

**Table 9**

Volatility of microstructure noise under different regimes.

This table shows the standard deviation of the microstructure noise,  $p_t - \hat{v}_t$ , where  $\hat{v}_t$  is the expectation of an uninformed trader about the fundamental value, given market observables and the lagged fundamental value  $v_{t-\Delta t}$ . We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). These one million entries form the data on which this table is based. For each transaction in the market, we construct the expectation of  $v_t$  for an uninformed agent. The estimated microstructure noise associated with that transaction is the transaction price minus this expectation  $\hat{v}_t$ , measured in ticks. The true microstructure noise is the transaction price minus the fundamental value  $v_t$ , also measured in ticks. The table reports the standard deviation of the microstructure noise across different volatility and information regimes.

Volatility regime	Information regime	Standard deviation	
		True noise ( $p_t - v_t$ )	Estimated noise ( $p_t - \hat{v}_t$ )
Low	All agents informed	0.79	0.68
	Speculators informed	1.25	0.79
High	All agents informed	1.02	0.88
	Speculators informed	1.68	0.90

dispersion on the sell side (depth away from the ask) tends to lead to higher prices.

## 5. Microstructure noise and implications

The price at which a transaction occurs in the market is frequently decomposed as

$$p_t = \hat{v}_t + n_t,$$

where  $p_t$  is the price at time  $t$ ,  $\hat{v}_t$  is the expected fundamental value given what the econometrician observes (also called the “efficient price”), and  $n_t$  is the deviation from the fundamental value or the “microstructure noise.”<sup>14</sup> In a frictionless market with all agents informed, if the econometrician can perfectly estimate the fundamental value, all trades should execute at the true fundamental value,  $v_t$ , and the microstructure noise should be identically zero. We show in this section that the properties of the microstructure noise vary with both the volatility of the fundamental value and the information regime in the market.

In estimating the microstructure noise, to be fair to the spirit of the analysis, we use the actual transaction price and the expected value of  $v$  given market observables and the lagged fundamental value. As noted earlier, this expected value is kept track of even when all agents are informed, thus allowing us to conduct the analysis in the role of uninformed market observers.

### 5.1. Volatility of microstructure noise

If the fundamental value were perfectly observable, and all trades occurred at the fundamental value, the microstructure noise would be identically zero. Thus, the standard deviation of the microstructure noise is a measure of the trading frictions present in the market in equilibrium.

The third column of Table 9 shows that, indeed, the volatility of the true microstructure noise (i.e., price minus true fundamental value) does increase with both fundamental volatility and with asymmetric information. However, as the last column of Table 9 shows, asymmetric information has a negligible to small effect on estimated volatility of the microstructure noise. As expected, excess volatility is somewhat higher when the volatility in the fundamental value is high, compared to when it is low.

**Observation 7.** *The market acts as a volatility multiplier: The volatility of the true microstructure noise increases when the volatility of the fundamental value is higher, and when only speculators are informed. However, the effects of asymmetric information and fundamental volatility on the volatility of the estimated microstructure noise are weaker.*

The intuition for the first part of Observation 7 partly follows from Observation 2. Under high volatility, speculators provide less liquidity, so quotes are more often set by agents with a large private value. These agents are willing to trade at different prices (specifically, prices closer to the fundamental value) than those provided by speculators, thus increasing the overall noise in the quotes. Since every transaction price is a quote before the transaction is consummated by an incoming market order, this translates into noisier transaction prices under high volatility. Similarly, with adverse selection, quotes are typically set further away from the fundamental value, leading to excess volatility in transaction prices.

The explanation for the latter part of Observation 7 lies in the fact that estimates of the fundamental value are themselves more noisy when only speculators are informed, as shown in Table 7. The true microstructure noise (that is, the difference between price and true fundamental value of the asset) is indeed more volatile when only speculators are informed. However, in some sense, that volatility is soaked up in noisy estimates of fundamental value, and the estimated volatility of the microstructure noise shows little variation across information regime. Indeed, while the estimated noise is more volatile when fundamental volatility is high, the effect there is

<sup>14</sup> Hasbrouck (2002) provides a comprehensive discussion of the decomposition of transaction price into efficient price and microstructure noise.

also dampened due to the estimate of fundamental value being more noisy.

In using the volatility of prices to proxy for the volatility of the fundamental value, the proportion of informed traders in the market must be taken into account. A low volatility asset with few informed traders and a high volatility asset with a larger proportion of informed traders may have similar price volatilities. Controlling for the proportion of informed traders, assets with high true volatility will have high price volatility as well, so sorting assets by price volatility will effectively sort them by true volatility. However, the relationship between the two need not be linear, so using price volatility directly in a regression may yield misleading results.

### 5.2. Correlations in changes in microstructure noise and fundamental value

In many econometric studies, the microstructure noise  $n_t$  is assumed to be white noise. In our simulated data,  $v_t$  is an exogenous random walk, so its first difference is stationary. Let  $\Delta(x_t) = x_t - x_{t-1}$ . Table 10 exhibits the correlations between  $\Delta(n_t)$  and  $\Delta(\hat{v}_t)$  at the transaction-by-transaction level. The correlations are negative across all regimes, and are slightly higher in magnitude (i.e., further away from zero) when the asset has high volatility, and when there is asymmetric information across traders.

Hansen and Lunde (2006) empirically find a negative correlation between the fundamental value and the level of microstructure noise in 2004 for most of the stocks in the Dow Jones Index. The Dow Jones stocks are, of course, not traded on a pure limit order market. Nevertheless, the NYSE and Nasdaq have both incorporated the notion of a limit order book as part of their market design. Some of the intuition of our model may therefore be expected to carry over. As they point out, if changes in the transaction price lag changes in the fundamental value, it is immediate that there will be a negative correlation between changes in the fundamental value  $v_t$  and levels of the microstructure noise  $n_t$ . Suppose the fundamental value increases by one tick. There will typically be a lag before the price adjusts to the new level. In the interim,

the microstructure noise must mechanically decrease, inducing a negative correlation between changes in the true fundamental value  $v_t$  and levels of  $n_t$ . However, this need not translate to a negative correlation between changes in the estimated fundamental value  $\hat{v}_t$  and the resultant microstructure noise. The latter relies on the learning in the market being rapid enough for changes in  $v$  to flow through relatively quickly.

A similar inertia in prices leads to positive autocorrelation in the microstructure noise—prices take some time to adjust to changes in the fundamental value. Thus, if the microstructure noise becomes negative after (say) an increase in the fundamental value, it stays negative for a few transactions. Ait-Sahalia, Mykland, and Zhang (2006) consider serial dependence in the microstructure noise under the assumption of independence between microstructure noise and fundamental value. For stocks such as Microsoft and Intel, they show that, assuming this independence, trade reversals imply a negative autocorrelation in the microstructure noise (as mentioned earlier, it should be observed that these stocks do not trade in a pure limit order market). However, controlling for trade reversals, the remaining autocorrelation is positive, perhaps due to a gradual adjustment in prices following a change in fundamental value.

As also reported in Table 10, in our model, the autocorrelation in the microstructure noise is significantly affected by asymmetric information. When fewer traders are informed, prices adjust more slowly to changes in the fundamental value. Under symmetric information (i.e., with all agents informed), the autocorrelation is weaker under high volatility, whereas it is approximately the same across volatility regimes when speculators have superior information.

We also consider the correlations between changes in the microstructure noise and changes in the fundamental value. As shown in Table 10, changes in  $\hat{v}_t$  and changes in  $n_t$  are negatively correlated. This correlation potentially leads to a misestimation of  $\beta$ 's for the asset in question (see Section 5.3).

**Observation 8.** *Changes in the microstructure noise are negatively correlated with changes in the fundamental value.*

**Table 10**

Correlations between changes in fundamental value and microstructure noise, transaction-by-transaction.

This table shows the correlations between the microstructure noise and changes in the fundamental value across different volatility and information regimes. We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). These one million entries form the data on which this table is based. For each transaction in the market, we construct the expectation of  $v_t$  for an uninformed agent,  $\hat{v}_t$ . The microstructure noise  $n_t$  associated with that transaction is the transaction price minus this expectation, measured in ticks. The table reports the correlation between changes in the fundamental value across transactions,  $\Delta(\hat{v}_t)$ , and the microstructure noise,  $n_t$ , as well as changes in the microstructure noise across transactions,  $\Delta(n_t)$ . The autocorrelation of the microstructure noise is also reported. The  $p$ -values for the reported correlation coefficients are less than 0.001 in each case.

Volatility regime	Information regime	Correlation between		Autocorrelation of $n_t$
		$\Delta(\hat{v}_t)$ and $n_t$	$\Delta(\hat{v}_t)$ and $\Delta(n_t)$	
Low	All agents informed	−0.05	−0.33	0.25
	Speculators informed	0.04	−0.17	0.34
High	All agents informed	−0.21	−0.43	0.04
	Speculators informed	−0.04	−0.43	0.13

The correlation is mildly stronger when the asset has high volatility and when there is asymmetric information across market participants.

The second statement in Observation 8 represents a new testable implication. The intuition behind the observation is as follows. Suppose there has been no change in the fundamental value  $\nu$  over some period of time. We expect transaction prices to be close to  $\nu$  in such a situation, so that the microstructure noise  $n$  is close to zero. Now, suppose  $\nu$  increases by one tick. A negative correlation between the change in  $\nu$  and the change in  $n$  implies that  $n$  is now negative. In other words, bid and ask quotes adjust slowly to changes in the fundamental value  $\nu$  (recall that every transaction price is a bid or ask price before a transaction is consummated). The magnitude of the correlation between  $\Delta(\nu)$  and  $\Delta(n)$  captures the degree of the inertia in transaction price.

When all agents are informed, quotes adjust more quickly to changes in the fundamental value. Therefore, the correlation between  $\Delta(\nu)$  and  $\Delta(n)$  is stronger when only speculators are informed. Comparing high volatility regimes to low volatility ones, changes in  $\nu$  are as frequent under both regimes, but of a higher magnitude when volatility is high. Thus, following a change in  $\nu$ , existing orders on the book are mispriced to a greater degree, and are more likely to be picked off by incoming traders. Finally, when there is high volatility and only speculators are informed, the correlation is further strengthened by the shift in liquidity provision at the margin, from informed speculators to uninformed large private value traders.

### 5.3. Quantifying the cross-sectional asset pricing implications

We next consider the impact of the correlation between microstructure noise and fundamental value on asset prices. Let  $i$  denote a specific asset, and let  $\Delta(\hat{\nu}_{it}) = \Delta(\nu_{it}) + e_{it}$ . Since the market's beliefs about the fundamental value are on average correct,  $e_{it}$  has mean zero. Next, suppose that the relationship between changes in the microstructure noise  $\Delta(n_{it})$  and the true value of the asset  $\Delta(\nu_{it})$  is linear so that we can write

$$\Delta(n_{it}) = \gamma_i \Delta(\nu_{it}) + u_{it},$$

where the residual  $u_{it}$  is uncorrelated with  $\nu_{it}$ . Note that  $\Delta(n_{it})$  and  $\Delta(\nu_{it})$  both have a mean of zero, and  $\gamma_i < 0$  due to the negative correlation between changes in the microstructure noise and the fundamental volatility.

Define the “true” return process for firm  $i$  as  $r_{it}^* = (\nu_{it} - \nu_{it-1})/\nu_{it} = \Delta(\nu_{it})/\nu_{it}$ , whereas the observed return process is

$$\begin{aligned} r_{it} &= \frac{\Delta(p_{it})}{p_{it}} = \frac{\Delta(\hat{\nu}_{it}) + \Delta(n_{it})}{\hat{\nu}_{it} + n_{it}} \\ &= (1 + \gamma_i)r_{it}^* \frac{\nu_{it}}{\hat{\nu}_{it} + n_{it}} + \frac{u_{it} + e_{it}}{\hat{\nu}_{it} + n_{it}}. \end{aligned}$$

Most of the transaction prices in our simulated data are within a few ticks of the fundamental value. That is,  $n_{it}$  is a few ticks for most transactions. Hence, for many stocks, it

follows that  $\nu_{it}/(\hat{\nu}_{it} + n_{it}) \approx 1$ , so that we can write  $r_{it} \approx (1 + \gamma_i)r_{it}^* + (u_{it} + e_{it})/(\hat{\nu}_{it} + n_{it})$ .

Now, suppose the Capital Asset Pricing Model (CAPM) holds, so that the true return process satisfies

$$r_{it}^* = r_f + \beta_i(r_{mt} - r_f) + \varepsilon_{it},$$

where  $r_f$  is the risk-free rate,  $r_{mt}$  the return on the market, and  $\varepsilon_{it}$  a mean-zero i.i.d. shock. The true  $\beta$  is defined by  $\beta_i = \text{cov}(r_{it}^*, r_{mt})/\sigma_{mt}^2$ . The  $\beta$  estimated from transaction data is  $\hat{\beta} = \text{cov}(r_{it}, r_{mt})/\sigma_{mt}^2 \approx (1 + \gamma_i)\beta$ .

Thus, correlation between changes in the microstructure noise and fundamental value can lead to a misestimation of  $\beta$ . To obtain a quantitative measure of the magnitude of possible misestimation, we regress  $\Delta(n_{it})$  on  $\Delta(\hat{\nu}_{it})$  in each of the four cases in our simulation. Note that this introduces another element of noise in the estimation: the fundamental value must be estimated as well.

In this regression, we use transaction-level data.<sup>15</sup> The results are reported in Table 11. The  $\gamma$  coefficients are negative in each case, are lower under asymmetric information (holding volatility fixed), and slightly higher under high volatility (holding information regime fixed). Since the coefficients are negative, the estimated beta coefficients will be too low, suggesting that the estimated risk premium may be too high.

A further implication is that if the estimated beta coefficients are too low to explain asset returns, other “factors” related to market outcomes, such as liquidity or idiosyncratic risk (which is related to the fundamental volatility of the asset), may appear significant in an asset pricing regression, even if the true model is the CAPM. Further, other microstructure variables that are correlated with microstructure noise, such as bid–ask spreads or other measures of transaction cost, should also be informative about observed returns. The intuition, of course, extends to multi-factor models as well.

Bessembinder and Kalcheva (2008) consider the effects of “bid–ask bounce,” also a symptom of microstructure frictions, on asset pricing tests, and find that not only are  $\beta$ 's and returns misestimated, but incorrect inferences may also be plausibly obtained in asset pricing tests. Further, they find that illiquidity can appear to be priced even though it is not in the true model. Interestingly, Bessembinder and Kalcheva (2008) find a bias in the opposite direction to ours, in a model in which the microstructure noise is independent and identically distributed over time. Our work therefore emphasizes the importance of controlling for the properties of the microstructure noise.

A related notion is that the degree of misestimation will vary based on the properties of the asset at hand. For example, small stocks tend to be more volatile in prices, and may be expected to have high fundamental volatility as well. Although agents have a greater incentive to acquire information about such stocks, the cost of acquiring this information is also greater, compared to a

<sup>15</sup> Bandi and Russell (2005) show that using high-frequency data to estimate asset betas results in inconsistent estimates, whereas using low-frequency data leads to imprecise ones, resulting in a tradeoff.

**Table 11**

Regression of change in microstructure noise on change in estimated fundamental value.

This table reports coefficient estimates and standard errors from an OLS regression of the change in the microstructure noise,  $\Delta(n_t)$ , on the change in the expectation of fundamental value,  $\Delta(\hat{v}_t)$ , for different volatility and information regimes. We numerically find the equilibrium for each information and volatility regime and then simulate a further one million entries into the market (including reentries by traders who have previously arrived at the market). These one million entries form the data on which this table is based. For each transaction in the market, we construct the expectation of  $v_t$  for an uninformed agent,  $\hat{v}_t$ . The microstructure noise  $n_t$  associated with that transaction is the transaction price minus this expectation, measured in ticks. The change in each of  $n_t$  and  $\hat{v}_t$  is the value at a given transaction minus the value at the previous transaction. The change in  $n_t$  is then regressed on a constant and the change in  $\hat{v}_t$ , and the slope coefficient  $\gamma$  from this regression is reported, with the standard error in parentheses. For each volatility and information regime, the last two columns contain the number of observations and the  $R^2$  of the regression.

Volatility regime	Information regime	Coefficient (std. error)	No. of obs.	$R^2$
Low	All agents informed	−0.55 (0.003)	200,242	0.11
	Speculators informed	−0.26 (0.003)	207,143	0.03
High	All agents informed	−0.51 (0.002)	213,594	0.18
	Speculators informed	−0.40 (0.002)	202,098	0.19

large, liquid stock. Therefore, asymmetric information is likely to be a more important friction for a small stock. To the extent that variables such as those related to liquidity or idiosyncratic risk capture the effects of a misestimation of factor betas, these variables will then appear to explain returns in a cross-section of assets as well.

## 6. Conclusion

We model agent behavior in a dynamic limit order market, allowing for the possibility of adverse selection in the form of asymmetric information across agents about the fundamental value of an asset. Agents with no intrinsic motive for trade (or speculators) have a greater incentive to acquire information about the asset. In addition, the sequential arrival of traders is itself a friction in a dynamic market, since newly arrived traders may have more recent information.

We show that agents' trading strategies change with the properties of the fundamental value and in the presence of adverse selection. On average, speculators set the bid and ask quotes when the fundamental value has low volatility. However, when fundamental volatility is high, speculators reduce their provision of liquidity, so that quotes are more often set by agents with high private values. As a result, there is an increase in the volatility of the microstructure noise. Agents with high private values are more willing to provide liquidity when the proportion of informed agents in the market is low. Thus, markets in which only speculators are informed exhibit even greater volatility in the transaction price.

Transaction prices may take some time to adjust to changes in the fundamental value. Further, their response depends on agents' trading strategies, which in turn depend on properties of the fundamental value and on the information regime. Changes in the microstructure noise are negatively correlated with changes in the estimated fundamental value of the asset. Therefore, any econometric specification that assumes such noise is uncorrelated with fundamental volatility misestimates factor exposure. This leaves open the possibility that other variables that are correlated with microstructure noise

(such as many transaction costs or liquidity measures) could be priced.

The microstructure noise itself exhibits positive serial correlation. Thus, an accurate decomposition of transaction price into estimated fundamental value and microstructure noise must take account of these properties of the microstructure noise.

Finally, if all assets are traded in limit order markets, microstructure noise may be an undiversifiable risk, and therefore priced in equilibrium. For example, [Bandi, Moise, and Russell \(2006\)](#) consider a three-factor model which includes the market, innovations in market volatility, and innovations in microstructure volatility on the market portfolio (proxied by the S&P 500 "spider" contracts), and show that innovations in microstructure volatility are priced in the cross-section.

Since our work suggests that microstructure noise varies across assets and is correlated (in changes) with the fundamental value, it is potentially important in a characteristics framework as well. For example, [Daniel and Titman \(1997\)](#) suggest that factor pricing models may be picking up differences in fundamental characteristics across (say) industries. Since the volatility of the fundamental value is likely to be closely related to other fundamental characteristics, one may expect microstructure noise to be priced as well.

## Appendix A

### A.1. Model description: trading game

Recall that the trading game assumes that traders' information acquisition decisions are fixed. Let  $I \in \{0, 1\}$  denote the action an agent takes with respect to information acquisition, where  $I = 1$  if the agent chooses to become informed. As mentioned in Section 2, the type of a trader is given by  $\theta = (\rho, \alpha)$ , where  $\rho$  is a discount rate and  $\alpha$  a private value for the asset. Let  $\Theta$  denote the set of feasible agent types, and  $\Theta^I$  the set of feasible  $(\theta, I)$  pairs.

Now, when he is in the market at time  $t$ , a trader takes an action  $a = (p, q, x)$ , where  $p$  denotes the price at which he submits an order,  $q \geq 0$  the priority of his order among

all orders at price  $p$ , and

$$x = \begin{cases} 1 & \text{if a buy order is submitted,} \\ -1 & \text{if a sell order is submitted,} \\ 0 & \text{if no order is submitted.} \end{cases} \quad (2)$$

Recall that  $L$  denotes the limit order book. For a newly submitted order,  $L$ ,  $p$ , and  $x$  determine  $q$ . However,  $q$  evolves over time for an order on the books, and may change before the trader reenters the market. It is used in determining the continuation payoff on reentry. If  $x = 0$ , the values of  $p$  and  $q$  are irrelevant to expected payoff.

If there is an existing order at price  $p$  on the other side of the market, a submitted order executes instantaneously and is called a market order. A buy order at a price  $p \geq A(L)$  automatically executes at the ask price  $A(L)$ , and similarly with a sell order at  $p \leq B(L)$ , where  $B(L)$  is the bid price. For such an order, we set  $q = 0$ . Alternatively, if there is no order on the other side of the market at that price, the order joins the existing orders on the same side at that price.

With a slight abuse of notation, let  $\ell^p$  denote the outstanding limit orders at price  $p$  before the agent submits an order. Then, for a new order, priority  $\hat{q}(p, x)$  is determined as follows:

$$\hat{q}(p, x) = \begin{cases} 0 & \text{if (i) } x = 0 \text{ or} \\ & \text{(ii) } x = 1, p \geq A(L) \text{ or} \\ & \text{(iii) } x = -1, p \leq B(L), \\ |\ell^p + x| & \text{otherwise.} \end{cases} \quad (3)$$

Consider the problem faced by a trader in the market at time  $t$ . Suppose this trader is reentering the market (the problem faced by a new trader is identical to the problem faced by a reentering trader who did not submit an order on his previous entry), and, on his previous entry (at some  $t' < t$ ), he had submitted an order at price  $p$  that is still active. This order may have improved in priority at price  $p$  between times  $t'$  and  $t$ . The trader has the option of leaving the order unchanged and taking no further action. The trader's optimal action depends on the state he observes.

Let  $s(\theta, I)$  be the state observed on a particular entry to the market at time  $t$  by a trader with type  $\theta$  who has taken the information acquisition action  $I$ . Here,  $s(\cdot)$  includes:

- (i) the history of play in the game, and the history of changes in  $v$  up to time  $t - \Delta_t$  (if  $I = 0$ ; i.e., the agent is uninformed) or time  $t$  (if  $I = 1$ ; i.e., the agent is informed). If the trader had previously entered the market and taken an action,  $s(\cdot)$  includes the status of the previous action,  $a = (p, q, x)$ , where  $p$  is the price at which the previous order was submitted,  $q$  its current priority at price  $p$ , and  $x$ , which is defined in Eq. (2) of the text to take on the value  $+1$  for a buy order,  $-1$  for a sell order, and zero if no action was taken. If the trader is entering the market for the first time,  $x$  is set to zero;
- (ii) a variable  $z \in \{0, 1\}$  that denotes the number of shares the agent has available to trade. Each trader enters with  $z = 1$ . Once he has traded,  $z$  is set to zero. As we comment after the Bellman equation in Eq. (6), this

variable is used to conveniently set an agent's future payoff to zero once he has traded.

Let  $\mathcal{A}(s)$  denote the feasible action set of a trader in state  $s$ . For computational tractability, we restrict limit order submission to a finite set of prices within  $k$  ticks of an agent's expectation of the fundamental value. We choose  $k$  to be sufficiently large that it does not affect the equilibrium. Denote the agent's expectation of fundamental value as  $\hat{v}(s) = E(v|s)$ , where  $s$  denotes the state observed by the agent. The feasible action set is then defined as

$$\mathcal{A}(s) = \{(p, q, x) \mid \text{(i) } x \in \{-1, 0, 1\}, \text{ (ii) } q = \hat{q}(p, x), \\ \text{(iii) } q \neq 0 \implies p \in [\hat{v}(s) - k, \hat{v}(s) + k] \cap \mathcal{P}\}. \quad (4)$$

Let  $S_{(\theta, I)}$  denote the set of feasible states a trader with type  $\theta$  and information  $I$  may encounter. A mixed strategy for such a trader in the trading game is then a map  $\sigma_{(\theta, I)} : S_{(\theta, I)} \rightarrow \prod_{s \in S_{(\theta, I)}} \Delta(\mathcal{A}(s))$ , where  $\Delta(\mathcal{A}(s))$  is the set of probability distributions over  $\mathcal{A}(s)$ .

Consider the value to trader type  $\theta$ , who takes information acquisition action  $I$ , of being in the state  $s$ , given that his previous order is  $a$ . Each action  $\tilde{a}$  in the finite action set  $\mathcal{A}(s)$  gives rise to an expected payoff that consists of two components: first, a payoff conditional on the order executing before the trader reenters the market, and second, the value associated with reentering the market (without having executed in the interim) in some new state  $s$ .

The likelihood of a limit order executing clearly depends on the strategies of other players in the game. Since we consider only symmetric equilibria, consider a trader in the market, and let  $\sigma = \{\sigma_{(\theta, I)}\}_{\theta \in \Theta^I}$  denote the strategy adopted by every other player. For convenience, normalize the trader's entry time to zero. Let  $\phi(\tau, v; s, \tilde{a}, \sigma)$  be the probability that an action  $\tilde{a} = (\tilde{p}, \tilde{q}, \tilde{x})$  taken in state  $s$  at time zero leads to execution at time  $\tau > 0$  when the fundamental value is  $v$ , given that all other agents are playing  $\sigma$ , and let  $f(v|s, t)$  denote the density function over  $v$  at time  $\tau$ , given state  $s$ . For an uninformed trader,  $f(\cdot)$  incorporates the trader's beliefs over  $v_0$ .

Suppose the trader reenters the market at some time  $w > 0$ . His expected payoff due to execution prior to reentry is

$$\pi(s, \tilde{a}, w, \sigma) = \int_0^w \int_{-\infty}^{\infty} (e^{-\rho t} \tilde{x}(\alpha + v_t - \tilde{p}) \phi(\cdot)) \times f(v|s, t) dv dt. \quad (5)$$

This equation is derived as follows. Suppose the agent's order executes at a time  $t \in [0, w]$ . The payoff to the order depends on the fundamental value at time  $t$ , which we denote  $v_t$ . As noted, the instantaneous payoff of this order at time  $t$  is  $\tilde{x}(\alpha + v_t - \tilde{p})$ . This payoff must then be discounted back to time zero, at the rate  $\rho$ . The innermost integral of the first term is over the different fundamental values that can obtain at time  $t$ . We expect  $\phi(\cdot)$  to be higher when  $v$  has moved in an adverse direction (for example,  $v$  has decreased after a limit buy was submitted)—this is another manifestation of adverse selection in this model. For a market order, we have  $\phi(0, \cdot) = 1$ ,

since the order executes immediately. The outermost integral is over the possible times at which execution could occur.

Recall that the reentry time is random and exogenous. Let  $G(\cdot)$  denote the probability distribution of the reentry time. Let  $v(s'|s, \tilde{a}, w, \sigma)$  denote the probability that the trader observes state  $s'$  on reentry, given action  $\tilde{a}$ , previous state  $s$ , elapsed time  $w$  since entry into the market, and strategy of other players  $\sigma$ . Finally, let  $J(s)$  denote the value to an agent of being in state  $s$ . The Bellman equation for the agent's problem is

$$J(s, \sigma) = \max_{\tilde{a} \in A(s)} \int_0^\infty \left\{ \pi(s, \tilde{a}, w, \sigma) + e^{-\rho w} \int_{s' \in S_\theta} J(s', \sigma) \times v(s'|s, \tilde{a}, w, \sigma) ds' \right\} dG(w). \quad (6)$$

The first term on the right-hand side (defined in (5)) indicates the payoff from execution before reentry at the random time  $w$ . The second term captures the continuation payoff to the trader on reentry to the market at time  $w$ . If his order executes before he reenters, we have  $z' = 0$  (i.e., he cannot trade any more shares). Define  $J(s', \sigma) = 0$  for all  $s'$  such that  $z' = 0$ , to ensure that the continuation payoff is set to zero if an order executes before the trader reenters the market.

The agent reenters the market at the random time  $w$  in some state  $s'$  different from  $s$ . If his previous order is still unexecuted, he can choose instead to submit a new order at a price  $\tilde{p} \neq p$ , and possibly in a direction  $\tilde{x} \neq x$ . A new order implies cancellation of the previous order. Alternatively, he can choose to leave his previous order on the books by setting  $\tilde{p} = p$  and  $\tilde{x} = x$ . Of course, market conditions may have changed since he first submitted the order, either due to exogenous reasons (e.g., a change in the fundamental value) or due to actions taken by other agents. The latter could enhance the priority of this agent's order at the price  $p$ , or it could reduce the overall priority if other agents submitted limit orders at prices more aggressive than  $p$ . Hence, the action  $a$  taken at time zero evolves to  $a'$  by the time the trader reenters at time  $w$ . The outermost integral is over the random reentry time.

Since the action set is finite on any entry, the maximum over all feasible actions exists and is well-defined. The value of a state and previous action pair is just the maximal expected payoff over all feasible actions the trader can take.

Fixing the strategies of all other agents, a given pure strategy  $y_{(\theta, I)}^*$  for a trader with type  $(\theta, I) \in \Theta^I$  is a best response if (and only if), for every  $s \in S_{(\theta, I)}$ ,

$$y_{(\theta, I)}^*(s) \in \arg \max_{\tilde{a} \in A(s)} \int_{w=0}^\infty \left\{ \pi(s, \tilde{a}, w, \sigma) + e^{-\rho w} \int_{s' \in S_\theta} J(s', \sigma) \times v(s'|s, \tilde{a}, w, \sigma) ds' \right\} dG(w). \quad (7)$$

Note that some of these states may not be attained in equilibrium. Nevertheless, we require the trader to act optimally in these states as well. Also, the trader's optimal action in any state must take into account the possibility

of future reentry (and that the trader will play optimally in the new state).

Finally, a strategy for each player is defined as  $y = \{y_{\theta, I}\}_{\theta \in \Theta^I}$ . A strategy  $y^* = \{y_{(\theta, I)}^*\}_{\theta \in \Theta^I}$  represents a Markov perfect Bayesian equilibrium of the trading game if, for each pair  $(\theta, I) \in \Theta^I$ ,  $y_{(\theta, I)}^*$  is a best response in every feasible state  $s \in S_{(\theta, I)}$ , given that all other agents are using the strategy  $y^*$ .

## A.2. Details of the numerical algorithm

We fix information acquisition strategies, and solve for the equilibrium of the corresponding trading game. We use an asynchronous value function iteration procedure, similar to Pakes and McGuire (2001), to find a  $J(s, \sigma)$  that satisfies the Bellman equation in Eq. (6). In principle, we would like agents to condition their strategies on the entire history of the game. In practice, of course, this is computationally infeasible. We therefore use the properties of the model to simplify the state space in the simulations.

Since the fundamental value evolves as a random walk, the set of prices at which trade can feasibly occur is, in principle, unbounded (although it is finite in any finite simulation). However, given the payoff on execution in Eq. (1) of the text, a trader cares only about the relative price at which trade occurs (i.e., the price relative to the fundamental value).

Historical prices and lagged values of  $v$  can also be expressed in terms of the current fundamental value for an informed trader. In the same manner, historical transaction prices and current books can be expressed relative to the last observed fundamental value for an uninformed trader. This significantly reduces the size of the state space, to the point that the set of recurrent states in our simulations is finite (although still very large).<sup>16</sup>

For the numerical implementation of the model, we restrict the state space for each agent as follows. Let  $m_t(I)$  denote the market conditions observed by an agent at time  $t$  (recall that informed agents, with  $I = 1$ , observe the current fundamental value; uninformed agents, with  $I = 0$ , observe it with a lag). We use

$$m_t(0) = \{\hat{L}_t, v_{(t-\Delta_t)}, \hat{p}_t, b_t\}, \\ m_t(1) = m_t(0) \cup \{v_t\},$$

where  $\hat{L}_t$  is a set of variables that depend on the book at time  $t$  ( $L_t$ ),  $\hat{p}_t$  is the price of the most recent transaction, and  $b_t$  is a variable indicating whether the most recent transaction was a buy ( $b_t = 1$ ) or sell ( $b_t = -1$ ).<sup>17</sup>

<sup>16</sup> A recurrent class is a subset of states with the following properties: (i) regardless of the initial state, the system eventually enters the recurrent class; (ii) once entered, the probability of each state outside the recurrent class is zero; and (iii) each state in the recurrent class is visited infinitely often as  $t$  approaches infinity.

<sup>17</sup> We investigated a model in which agents also observe the cumulative market buys and sells in the interval  $[t - \Delta_t, t]$ . The added conditioning variables are virtually ignored by traders in updating beliefs about  $v$ , and do not affect market outcomes. In our model, only recent history is relevant to traders, for two reasons. First, traders leave the market forever after execution. Therefore, any knowledge about traders who have already executed does not affect agents' beliefs about future

Ideally, we would like agents to condition on the entire book; in practice, this becomes computationally intractable. Instead, the variables in  $\tilde{L}_t$  consist of: (i) the current bid and ask prices,  $(B_t, A_t)$ ; (ii) the total depths at these prices,  $(\ell_t^B, \ell_t^A)$ ; and (iii) the cumulative buy and sell depths in the book,  $D_t^B = \sum_{i=0}^N \{\ell_i^t > 0\}$  and  $D_t^S = \sum_{i=0}^N \{\ell_i^t < 0\}$ .

Given the market observables used, in the numerical algorithm, the state at time  $t$  for an agent of type  $\theta$  who makes the information acquisition decision  $l$  is defined by  $s = \{\theta, m_t(l), a, z\}$ , where  $a = (p, q, x)$  denotes the status of his previous order, and  $z \in \{0, 1\}$  denotes how many shares he can trade ( $z$  is set to zero once he has traded).

In the algorithm, at each time  $t$ , each action  $\tilde{a}$  in each state  $s$  encountered by the simulation has an associated payoff  $U_t(\tilde{a}|s)$ . This payoff is a real number, and is the expected discounted payoff from taking action  $\tilde{a}$  in state  $s$ . Hence, it may be interpreted as the current belief of an agent about the payoff from this action.<sup>18</sup>

At any point of time, current beliefs  $U_t(\cdot)$  imply an optimal strategy profile  $y_t$ , which assigns the payoff-maximizing action in each state. Let  $\tilde{a}^*(s) \in \operatorname{argmax}_{\tilde{a} \in \mathcal{A}(s)} U_t(\tilde{a}|s)$  denote the optimal action in state  $s$ . Then, given beliefs  $U_t(\cdot)$ , the value of state  $s$  is determined as  $J(s, y_t) = U_t(\tilde{a}^*(s)|s)$ .

Each action and state pair  $(\tilde{a}, s)$  has an initial belief  $U_0(\tilde{a}|s)$ . These initial beliefs are set as follows. Consider a limit buy order at price  $p$  when the last observed fundamental value is  $v$ . The initial belief for such an order is the payoff  $\alpha + v - p$  discounted by the expected time until the arrival of a new trader for whom being a counterparty yields a non-negative payoff. This initial value is optimistic since (i) limit orders tend to execute when the fundamental value moves in an adverse direction, and (ii) counterparties usually hold out for a strictly positive payoff. The initial belief for market orders also assumes the fundamental value is unchanged from its last observed value, but of course involves no discounting. Given that we allow traders to tremble, any  $U_0(\cdot)$  can eventually lead to an equilibrium. The choice of initial beliefs is driven more by computational considerations (in particular, converging to equilibrium more quickly) than by a theoretical need.

Additional details of the algorithm are as follows.

1. Three types of exogenous events drive the simulation—changes in the fundamental value, the arrival of new traders, and the reentry of old traders who have not yet executed. At each point in time, let  $t_v$  denote the additional time until  $v$  changes,  $t_n$  the additional time until a new trader arrives, and  $t_r$  a vector of additional times until each old trader returns to the

market to possibly revise his order. Let  $\underline{t}_r = \min\{t_r\}$  denote the earliest reentry time across all old traders. Whenever an event occurs, we redraw the time until its next occurrence accordingly (recall that the time interval between events for a Poisson process has an exponential distribution). We also subtract the elapsed time from the other “time until” variables.

At time zero, we start with an empty book, new draws for  $t_v$  and  $t_n$ , and no existing traders (i.e.,  $t_r$  is an empty vector).

In theory, the initial fundamental value can be chosen arbitrarily. However, since  $v$  follows a random walk, the price grid would need to be infinite. To avoid this problem, the algorithm records all prices relative to the current  $v$ , and appropriately shifts all orders on the book whenever  $v$  changes.<sup>19</sup> The number of ticks around  $v$  for which orders are tracked is chosen sufficiently high that orders never “fall off” the grid. That is, orders get revised by returning traders before becoming too unaggressive for the grid, or get picked off before becoming too aggressive for the grid. We use an odd number of ticks, with  $v_t$  itself lying on a tick at all times.

2. At time  $t = \min\{t_v, t_n, \underline{t}_r\}$ , an exogenous event occurs. Suppose  $t_v < t_n$  and  $t_v < \underline{t}_r$ . Then, the fundamental value changes at time  $t_v$ ; with probability  $\frac{1}{2}$  it increases by one tick, and with probability  $\frac{1}{2}$  it decreases by one tick. As specified in item 1 above, we adjust the times for the three events as follows. We set  $t_n = t_n - t_v$  and  $t_r = t_r - t_v$ , and then draw a new time  $t_v$  for the next change in  $v$ .

Suppose, instead,  $t_n < t_v$  and  $t_v < \underline{t}_r$ . A new trader arrives to the market. His type is denoted as  $\theta = \{\rho, \alpha\}$ . The discount factor  $\rho$  is the same for all traders, and  $\alpha$  is drawn independently from the distribution  $F_\alpha$ . The times for the three events are adjusted as specified in item 1.

A given trading game is used to obtain payoffs to either equilibrium strategies or to deviator strategies in the information acquisition game. When obtaining equilibrium payoffs, we set  $l = \sigma_l(\theta)$  for the new trader. When obtaining payoffs to deviating, we classify a trader as a deviator with probability 0.02, as long as no other deviators are currently in the market (to preserve the notion of unilateral deviation). If the new trader is a regular trader, we set  $l = \sigma_l(\theta)$ . If he is a deviator, we set  $l = 1$  when  $\sigma_l(\theta) = 0$  and  $l = 0$  when  $\sigma_l(\theta) = 1$  (i.e., a deviator acquires information only when regular traders of his type do not). Importantly, beliefs and trading strategies of non-deviators are held fixed throughout the algorithm when obtaining payoffs to deviating in the information acquisition game.

Since the trader is new, we set  $z$  to one and his previous action  $x$  to zero. The trader observes the state  $s = \{\theta, m(l), a, z\}$  and takes an action  $\tilde{a}$ . If he submits a market order, he executes and leaves the market

(footnote continued)

play in the game. Second, for uninformed agents, events prior to  $t - \Delta t$  offer no information about changes in  $v$  since it was last observed (at time  $t - \Delta t$ ). Nevertheless, any particular snapshot of history is potentially restrictive. Computational reasons require us to impose such a restriction; without it, the state space is too large.

<sup>18</sup> Our  $U(\cdot)$  corresponds to the  $Q$  function in the  $Q$ -learning literature begun by Watkins (1989).  $Q$ -learning and other neuro-dynamic programming techniques related to our simulation algorithm are described in Bertsekas and Tsitsiklis (1996).

<sup>19</sup> Importantly, uninformed traders observe the prices of orders on the book relative to  $v_{t-\Delta t}$ , else they could directly infer  $v_t$  as the mid-tick in the book. That is, the algorithm tracks the book relative to  $v_t$  but presents it to traders relative to their last observed  $v$ .

forever. If he takes any other action, we draw his random return time and include it in the vector  $t_r$ . We also draw a new random time  $t_n$  before the arrival of the next new trader.

Finally, suppose  $t_r < t_v$  and  $t_r < t_n$ . An old trader returns to the market. He observes the current state  $s = \{\theta, m(I), a, z\}$  which includes the current status  $a$  of his previous action. He then takes some action (which could include retaining his previous order). If he submits a market order, he executes and leaves the market forever. If he takes any other action, we draw his new return time in  $t_r$ , and adjust the times  $t_n$  and  $t_v$  as specified.

- Suppose a trader of type  $\theta$  is in the market at time  $t$ . The trader observes the current state  $s = \{\theta, m(I), a, z\}$  and chooses a payoff-maximal action  $\tilde{a}^*(s) \in \arg\max_{\tilde{a} \in \mathcal{A}(s)} U_t(\tilde{a}|s)$ . If the trader is informed, he knows  $v_t$ , which determines  $\mathcal{A}(s)$ . If he is uninformed, his belief about  $v_t$  is used to determine  $\mathcal{A}(s)$ . Denote this belief as  $E(v_t|m_t(0))$ .

Beliefs about the current fundamental value are updated in the following manner. Let  $\delta_t(m_t(0)) = E(v|m_t(0)) - v_{t-\Delta_t}$  denote the extent by which an uninformed agent at time  $t$  revises his belief about  $v_t$ , given a lagged value  $v_{t-\Delta_t}$ . Since we consider stationary equilibria, we drop the time subscript on market conditions. Start with an initial belief  $\delta_0 = 0$  for each market  $m(0)$ . Let  $r(m(0))$  be an integer denoting the number of times market conditions  $m(0)$  are encountered in the simulation. We drop the argument of  $r$  for notational convenience. Each time a trader observes market conditions  $m(0)$ , we increment  $r$  by one, and set

$$\delta_r(m(0)) = \frac{r-1}{r} \delta_{r-1}(m(0)) + \frac{1}{r} (v_t - v_{t-\Delta_t}). \quad (8)$$

Recall that  $\hat{v}(m) = E(v|m)$  denotes a trader's estimate of the fundamental value. For an uninformed trader who enters in market  $m(0)$ , this estimate is  $\hat{v}(m(0)) = v_{t-\Delta_t} + \delta_{r-1}(m(0))$ , since the market  $m(0)$  has been observed  $r-1$  times prior to his entry.<sup>20</sup> Using this estimate  $\hat{v}$ , the action set for each trader is defined as in Eq. (4) of the text. Now, suppose the optimal action  $\tilde{a}^*$  does not represent a market order; that is, it is either a limit order or no order. Suppose further that, at some future point of time,  $t'$ , the trader reenters the market. He finds that his action has evolved to  $\tilde{a}'$ , and the new market is  $m'$ . Denote  $s' = \{\theta, m'(I), \tilde{a}', z\}$ .

The action  $\tilde{a}^*$  thus generates a realized continuation value  $J(s', y_{t'})$  on this visit, which is “averaged in” to the belief  $U_t(\tilde{a}^*|s)$  in the following manner. We define

$$U_{t'}(\tilde{a}^*|s) = \frac{n}{n+1} U_t(\tilde{a}^*|s) + \frac{1}{n+1} e^{-\rho(t'-t)} J(s', y_{t'}). \quad (9)$$

Here,  $n(\tilde{a}^*, s)$  is a positive integer that is incremented by one each time action  $\tilde{a}^*$  is chosen in state  $s$  (for notational brevity, the dependence of  $n$  on  $\tilde{a}^*$  and  $s$  is suppressed in Eq. (9)). We start with an initial positive integer  $n_0$  for each action and state pair  $(\tilde{a}, s)$ . This integer affects the speed at which the algorithm converges, with larger values implying slower convergence. Periodically, during the simulation, we reset  $n$  to  $n_0$  for some action and state pairs to obtain quicker convergence.

Similarly, suppose a trader submits a limit order (denoted by action  $\tilde{a}^*$ ) at time  $t$ , and this order executes against a market order submitted by another trader at time  $t'$ . The actual payoff to that limit order in the simulation is  $\tilde{\chi}(\alpha + v_{t'} - \tilde{p}^*)$ , where  $\alpha$  denotes the private value of the trader. In this case, we update

$$U_{t'}(\tilde{a}^*|s) = \frac{n}{n+1} U_t(\tilde{a}^*|s) + \frac{1}{n+1} e^{-\rho(t'-t)} \tilde{\chi}(\alpha + v_{t'} - \tilde{p}^*). \quad (10)$$

- Whenever a trader takes an action, his belief about the payoff to a market order is updated in similar fashion. For example, let  $\tilde{a}_b$  denote the action that involves submitting a market buy order, given market  $m$  and previous action  $a$ . In the simulation, we (as modelers) know the payoff to a market order in every state, whether a trader is informed about the current value of  $v$  or not. Hence, these payoffs can be averaged in for market orders even when such orders are suboptimal for the trader. For this updating, we use Eq. (10), with  $t' = t$  and  $v_{t'} = v_t$ .
5. In the simulation, most traders take the optimal action given current beliefs. If all traders did this, there is the possibility that the algorithm would be “stuck” at a non-equilibrium state—every trader of a given type would take the same action in that state, so these traders would never learn the payoffs to other actions in that state. If there is an error in beliefs, all traders of that type may play suboptimally.

To ensure that beliefs are updated for all actions in every state, we introduce trembles. Specifically, with probability  $\varepsilon$  a trader trembles over all suboptimal limit orders available to him. He chooses among suboptimal limit orders with equal probability. The algorithm will then naturally update the beliefs about payoffs to this action.<sup>21</sup>

<sup>20</sup> Note that the updating process does not use information about the agent's type. Hence, this updating can be (and is) performed even when the trader in the market is informed about the current value of  $v$ . This allows us to construct the beliefs over  $v$  of a hypothetical uninformed trader even when all traders in the market are informed ones.

<sup>21</sup> When a player trembles at  $t' > t$ , the payoff of the optimal action at  $t'$  is used to update  $U_t^k(\cdot)$  to  $U_{t'}^{k+1}(\cdot)$ . Thus, traders do not anticipate behaving suboptimally in the future.

### A.3. Convergence criteria

We run the model for a few billion events until we check for convergence. Along the way, we evaluate the change in value functions every 300 million new trader arrivals, by computing  $|U_{t_2}^{k_2}(\tilde{a}|s) - U_{t_1}^{k_1}(\tilde{a}|s)|$  for each pair  $(\tilde{a}, s)$  that occurs along the path of play in the simulation. Here,  $k_1$  is the number of times the action  $\tilde{a}$  has been chosen in state  $s$  at the start of the current 300 million new trader arrivals, and  $k_2 \geq k_1$  the number of times it has been chosen at the end of the current 300 million new trader arrivals. Further,  $t_1$  and  $t_2$  represent the actual time at the start and end of the 100 million arrivals.

Essentially, if this weighted absolute difference (weighted by  $k_2 - k_1$ ) is small, that suggests the value functions have converged. When this weighted difference is below 0.01, we apply other convergence tests. At this point, we hold the beliefs  $U(\cdot)$  fixed and simulate the model for a total 300 million more new trader arrivals (new and returning). Let  $U^*(\tilde{a}|s)$  be the fixed beliefs. These imply an optimal strategy profile  $y^*$ . For each  $(\tilde{a}, s)$ , define  $J(s, y^*) = \max_{\tilde{a} \in \mathcal{A}(s)} U^*(\tilde{a}|s)$ .

We compare the empirical payoffs from different actions in the simulation to the fixed beliefs. This comparison is done at two levels. The first is a “one-step ahead” check based on the trader’s next entry time or execution time, whichever is sooner. Suppose a trader takes an action  $\tilde{a}$  at time  $t$ , and reenters at  $t' > t$  with a new state  $s'$ . His one-step ahead empirical payoff is taken to be  $\tilde{J}_1(s, y^*) = e^{-\rho(t'-t)} J^*(s', y^*)$ . If the trader takes an action  $\tilde{a}$  at  $t$  and executes at  $t' > t$  before he can reenter, his one-step ahead empirical payoff is  $\tilde{J}_1(s, y^*) = e^{-\rho(t'-t)} \tilde{\chi}(\alpha + \nu_{t'} - \tilde{p})$ .

Second, eventually every trader in this model executes, and leaves the market. At the time he executes, he obtains a realized payoff. Suppose the trader enters at  $t$ , and eventually executes at  $t'$ . Let  $\tilde{a}$  denote his most recent action before execution. His realized payoff is then  $\tilde{J}(s, y^*) = e^{-\rho(t'-t)} \tilde{\chi}(\alpha + \nu_{t'} - \tilde{p})$ .

We use two convergence criteria for each of the two comparisons above, similar to those proposed by Pakes and McGuire (2001), and require that a simulation satisfy both criteria before it has converged. First, we consider the correlation between beliefs  $J^*(\cdot)$  and realized outcomes  $\tilde{J}$  or  $\tilde{J}_1$ , and require it to exceed 0.99 (for each of our simulations, in practice, once the second criterion is satisfied this correlation exceeds 0.999). Second, we consider the mean absolute error in beliefs, weighted by the number of times the state and action are observed, and require it to be less than 0.01.

## References

Admati, A., Pfleiderer, P., 1987. Viable allocations of information in financial markets. *Journal of Economic Theory* 43, 76–115.

Ait-Sahalia, Y., Mykland, P., Zhang, L., 2006. Ultra high frequency volatility estimation with dependent microstructure noise. Working paper, Princeton University.

Back, K., Baruch, S., 2007. Working orders in limit-order markets and floor exchanges. *Journal of Finance* 62, 1589–1621.

Back, K., Cao, H., Willard, G., 2000. Imperfect competition among informed traders. *Journal of Finance* 55, 2117–2155.

Bandi, F., Moise, C., Russell, J., 2006. Market volatility, market frictions, and the cross-section of stock returns. Working paper, University of Chicago.

Bandi, F., Russell, J., 2005. Realized covariance, realized beta, and microstructure noise. Working Paper, University of Chicago.

Barlevy, G., Veronesi, P., 2000. Information acquisition in financial markets. *Review of Economic Studies* 67, 79–90.

Bernardo, A., Judd, K., 1997. Efficiency of asset markets with asymmetric information. Working paper, UCLA Anderson, #16–97.

Bertsekas, D., Tsitsiklis, J., 1996. *Neuro-Dynamic Programming*. Athena Scientific, Belmont, MA.

Bessembinder, H., Kalcheva, I., 2008. Liquidity biases in asset pricing tests. Working Paper, University of Utah.

Biais, B., Bisiere, C., Spatt, C., 2003. Imperfect competition in financial markets: ISLAND vs. NASDAQ. Working paper, Carnegie Mellon University.

Bloomfield, R., O'Hara, M., Saar, G., 2005. The ‘make or take’ decision in an electronic market: evidence on the evolution of liquidity. *Journal of Financial Economics* 75, 165–199.

Boulatov, A., George, T., 2008. Securities trading when liquidity providers are informed. Working paper, University of Houston.

Chakravarty, S., Holden, C., 1995. An integrated model of market and limit orders. *Journal of Financial Intermediation* 4, 213–241.

Daniel, K., Titman, S., 1997. Characteristics or covariances? *Journal of Finance* 52, 1–33.

Diebold, F., Strasser, G., 2007. On the correlation structure of microstructure noise in theory and practice. Working paper, University of Pennsylvania.

Engle, R., Sun, Z., 2007. When is noise not noise—a microstructure estimate of realized volatility. Working paper, New York University.

Foster, S., Viswanathan, S., 1996. Strategic trading when agents forecast the forecasts of others. *Journal of Finance* 51, 1437–1478.

Foucault, T., 1999. Order flow composition and trading costs in a dynamic limit order market. *Journal of Financial Markets* 2, 99–134.

Foucault, T., Kadan, O., Kandel, E., 2005. The limit order book as a market for liquidity. *Review of Financial Studies* 18, 1171–1217.

Goettler, R., Parlour, C., Rajan, U., 2005. Equilibrium in a dynamic limit order market. *Journal of Finance* 60, 2149–2192.

Grossman, S., Stiglitz, J., 1980. On the impossibility of informationally efficient markets. *American Economic Review* 70, 393–408.

Hakansson, N., Kunkel, J., Ohlson, J., 1982. Sufficient and necessary conditions for information to have social value in pure exchange. *Journal of Finance* 37, 1169–1181.

Hansen, P., Lunde, A., 2006. Realized variance and microstructure noise. *Journal of Business Economics and Statistics* 24, 127–218.

Hasbrouck, J., 2002. Stalking the “efficient price” in market microstructure specifications: an overview. *Journal of Financial Markets* 5, 329–339.

Hirshleifer, J., 1971. The private and social value of information and the reward to inventive activity. *American Economic Review* 61, 561–574.

Hollifield, B., Miller, R., Sandás, P., Slive, J., 2006. Estimating the gains from trade in limit order markets. *Journal of Finance* 61, 2753–2804.

Jackson, M., 1991. Equilibrium, price formation, and the value of private information. *Review of Financial Studies* 4, 1–16.

Kaniel, R., Liu, H., 2006. So what orders do informed traders use? *Journal of Business* 79, 1867–1913.

Kumar, P., Seppi, D., 1994. Limit and market orders with optimizing traders. Working paper, Carnegie Mellon University.

Kyle, A., 1985. Continuous auctions and insider trading. *Econometrica* 53, 1315–1336.

Maskin, E., Tirole, J., 2001. Markov perfect equilibrium I: observable actions. *Journal of Economic Theory* 100, 191–219.

Mendelson, H., Tunca, T., 2004. Strategic trading, liquidity, and information acquisition. *Review of Financial Studies* 17, 295–337.

Pakes, A., McGuire, P., 2001. Stochastic algorithms, symmetric Markov perfect equilibrium, and the ‘curse’ of dimensionality. *Econometrica* 69, 1261–1281.

Parlour, C., 1998. Price dynamics in limit order markets. *Review of Financial Studies* 11, 789–816.

Radner, R., Stiglitz, J., 1984. A nonconcavity in the value of information. In: Boyer, M., Kihlstrom, R. (Eds.), *Bayesian Models in Economic Theory*. Elsevier Science Publishers, New York, pp. 33–52.

Rindi, B., 2008. Informed traders as liquidity providers: anonymity, liquidity and price formation. *Review of Finance* 12, 497–532.

Rosu, I., 2007. A dynamic model of the limit order book. Working paper, University of Chicago.

Spiegel, M., Subrahmanyam, A., 1992. Informed speculation and hedging in a noncompetitive securities market. *Review of Financial Studies* 5, 307–329.

Verrecchia, R., 1982. Information acquisition in a noisy rational expectations economy. *Econometrica* 50, 1415–1430.

Watkins, C., 1989. Learning from delayed rewards. Ph.D. Thesis, Cambridge University.